The Method of Singular Integral Equation in Problems of Liquid Oscillations in Coaxial Shells

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Abstract. The paper deals with problems of free vibrations of an ideal incompressible fluid in coaxial shells of revolution. It is assumed that the motion of the fluid is irrotational that allows us to introduce a velocity potential. In these suppositions the potential is satisfied to the Laplace equation. Boundary conditions are formulated on wetted surfaces of the shells and on a free liquid surface. The non-penetration conditions are applied to the wetted surfaces. On the free surface we consider dynamical and kinematical boundary conditions. The dynamical condition consists in equality of the liquid pressure on the free surface to the atmospheric one. The kinematic condition requires that total time derivative of the free surface elevation will be equal to zero at any instant. Regarding the potential of velocities, a boundary value problem is formulated that is further reduced to an eigenvalue problem. To solve the boundary value problem for the Laplace equation, boundary element methods are used in a direct formulation.

Keywords: singular integral equations, numerical solution, coaxial shells, splashing fluid, free vibrations.

1. INTRODUCTION

Coaxial shells of revolution, partially filled with liquid, are widely used as structural elements in various engineering applications, for example, in the petrochemical and nuclear industries. Piping systems can also be modelled with coaxial shells between which liquid fluid moves [1-3]. The application of analytical methods to study the vibrations of such systems is possible only for a relatively small class of coaxial shells. Therefore, many problems of calculating the frequencies and modes of fluid oscillations in coaxial shells remain unresolved and require the development of modern effective numerical methods. In this paper, to calculate the frequencies and forms of fluid free oscillations in rigid coaxial shells revolution, the boundary element method is used. We used previously developed methods for solving singular integral equations that arise in problems on vibrations of shells partially filled with fluid [4-6]. In [4], free and forced vibrations of elastic shells of revolution with liquids are studied; in [5], the case of the action of a seismic load is considered; in [6], large amplitudes of the external action are investigated, which led to the appearance of a chaotic nature of oscillations. The aim of this work is to generalize the methods of boundary integral equations and discrete singularities to determine the frequencies and forms of free oscillations of a liquid in rigid coaxial shells.

2. THEORETICAL BASIS

Two coaxial rigid shells are considered. The area between the shells can be completely or partially filled with an ideal incompressible fluid, Fig. 1.

It is required to find the frequencies and vibration modes of the fluid filling the region between the shells. It is assumed that the fluid motion is vortex-free. Under these conditions, there exists a velocity potential Φ satisfying the Laplace equation everywhere inside the region occupied by the liquid. The fluid pressure p on the wetted surfaces of the shell system is determined from the linearized Bernoulli integral

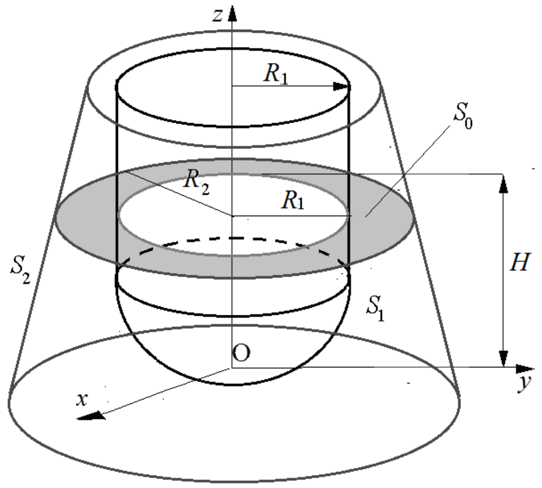

Fig. 1. Coaxial shells containing liquid.

$$p - p_0 = -\rho_l\left(\frac{\partial\Phi}{\partial t} + gz\right) \qquad (1)$$

Here g is the acceleration of gravity, ρ_l is the density of the fluid, p_0 is atmospheric pressure. To solve the Laplace equation, we define boundary conditions. On the wetted surfaces of the coaxial shells, S_1 and S_2 we require the fulfilment of the non-leakage condition

$$\left.\frac{\partial\Phi}{\partial n}\right|_{S_1} = 0 \qquad \left.\frac{\partial\Phi}{\partial n}\right|_{S_2} = 0 \qquad (2)$$

On the free surface S_0 we define the kinematic and dynamic conditions. The dynamic condition is the equality of the liquid pressure on the free surface to atmospheric pressure p_0 and the kinematic condition is the requirement that the total time derivative of the function describing the level of free surface rise be equal to zero

$$p - p_0 = 0\big|_{S_0}, \qquad \left.\frac{\partial\Phi}{\partial\mathbf{n}} = \frac{\partial\zeta}{dt}\right|_{S_0} \qquad (3)$$

Here, the function ζ describes the shape of the free surface and its position. Note that on the free surface the dynamic condition takes the following form:

$$\frac{\partial\Phi}{\partial t} + g\zeta = 0 \qquad (4)$$

We differentiate relation (4) with respect to t and substitute the resulting equality into the second of relations (3). We arrive at the following boundary value problem with respect to the unknown velocity potential Φ

$$\Delta\Phi = 0, \ \left.\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial\mathbf{n}} = 0\right|_{S_0}, \ \left.\frac{\partial\Phi}{\partial\mathbf{n}} = 0\right|_{S_1,S_2} \qquad (5)$$

For the unique solvability of the boundary value problem (5), we require the fulfilment of the Neumann condition

$$\int_{S_0}\frac{\partial\Phi}{\partial\mathbf{n}}dS_0 = 0 \qquad (6)$$

Under the assumption that the problem of small fluid vibrations is considered, the unknown velocity potential is presented in the form

$$\Phi(r,t) = \varphi(r)e^{i\omega}, \ i^2 = -1, \ r = (x,y,z) \qquad (7)$$

We come to the problem of eigenvalues

$$\Phi = 0, \ \left.\frac{\partial\Phi}{\partial\mathbf{n}} = \frac{\omega^2}{g}\Phi\right|_{S_0}, \ \left.\frac{\partial\Phi}{\partial\mathbf{n}}\right|_{S_1,S_2} = 0 \qquad (8)$$

Here ω is the frequency the natural oscillations of the liquid in the system of coaxial shells.

3. DATA

In the region occupied by the liquid and bounded by the surfaces S_0, S_1 and S_2 to determine the harmonic function Φ, we use the following integral representation [7]:

$$2\pi\varphi(P_0) = \iint_S \frac{\partial\Phi}{\partial\mathbf{n}}\frac{1}{|P - P_0|}dS - \iint_S \varphi\frac{\partial}{\partial\mathbf{n}}\frac{1}{|P - P_0|}dS \qquad (9)$$

$$S = S_0 \cup S_1 \cup S_2$$

Here $|P - P_0|$ is the Cartesian distance between the points P and P_0 located on the boundary S of the region occupied by the liquid, $\mathbf{n}$ is the unit vector of the external normal to the surface S.

It was shown in [8] that, using the integral representation (9), the boundary-value problem (8) that describes the eigenvalue problem, reduces to the following system of singular integral equations:

$$\begin{cases} 2\pi\varphi + \iint\limits_{S_1\cup S_2} \varphi \dfrac{\partial}{\partial \mathbf{n}}\left(\dfrac{1}{|P-P_0|}\right)dS \\ -\dfrac{\chi^2}{g}\iint\limits_{S_0}\varphi_0 \dfrac{1}{|P-P_0|}dS \\ +\iint\limits_{S_0}\varphi_0 \dfrac{\partial}{\partial z}\left(\dfrac{1}{|P-P_0|}\right)dS = 0, \\ -\iint\limits_{S_1\cup S_2} \varphi \dfrac{\partial}{\partial \mathbf{n}}\left(\dfrac{1}{|P-P_0|}\right)dS - 2\pi\varphi_0 \\ +\dfrac{\chi^2}{g}\iint\limits_{S_0}\varphi_0 \dfrac{1}{|P-P_0|}dS = 0, \end{cases} \tag{10}$$

In equations (10), φ denotes unknown values of the potential on the surfaces S_1 and S_2, and φ_0 denotes the values of the potential on the free surface S_0.

Further, similarly to [9], we introduce the integral operators

$$\mathbf{A}\varphi = 2\pi\mathbf{I}\varphi + \iint\limits_{S_1\cup S_2} \varphi \frac{\partial}{\partial \mathbf{n}} \frac{1}{r(P,P_0)}dS$$

$$\mathbf{B}\varphi_0 = \iint\limits_{S_0}\varphi_0 \frac{1}{r}dS$$

$$\mathbf{C}\varphi_0 = \iint\limits_{S_0}\varphi_0 \frac{\partial}{\partial z}\left(\frac{1}{r}\right)dS \tag{11}$$

$$\mathbf{D}\varphi = -\iint\limits_{S_1\cup S_1} \varphi \frac{\partial}{\partial \mathbf{n}} \frac{1}{|P,P_0|}dS$$

$$\mathbf{F}\varphi_0 = \iint\limits_{S_0}\varphi_0 \frac{1}{r}dS$$

Using (11), we reduce the boundary-value problem (8) to the following operator form:

$$\mathbf{A}\varphi = \frac{\chi^2}{g}\mathbf{B}\varphi_0 - \mathbf{C}\varphi_0$$

$$P_0 \in S_1\cup S_2 \quad , \quad P_0\in S_0 \tag{12}$$

$$\mathbf{D}\varphi = 2\pi\mathbf{I}\varphi_0 - \frac{\chi^2}{g}\mathbf{F}\varphi_0$$

Eliminating the function φ from equations (12), we obtain the eigenvalue problem in operator form, and only the values of the potential φ_0 on the free surface are unknown

$$\left(\mathbf{DA}^{-1}\mathbf{C} + 2\pi\mathbf{I}\right)\gamma_0 - \lambda\left(\mathbf{DA}^{-1}\mathbf{B}+\mathbf{F}\right)\varphi_0 = 0$$

$$\lambda = \frac{\chi^2}{g} \tag{13}$$

The eigenvalues and eigenvectors of the eigenvalue problem (13) are the frequencies and modes of liquid free oscillations in the system of coaxial shells.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be the generatrix of the composed shell of revolution. In [8-9] the following relationships are obtained:

$$\iint\limits_{S_1\cup S_1} \varphi \frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{|P-P_0|}\right)dS = \int\limits_{\Gamma}\varphi(z)\Theta(z,z_0)r(z)d\Gamma$$

$$\iint\limits_{S_0}\varphi_0 \frac{1}{|P-P_0|}dS_0 = \int\limits_0^R \varphi_0(r)\Phi(P,P_0)rdr$$

$$\Theta(z,z_0) = \frac{4}{\sqrt{a+b}}\frac{1}{2r}$$

$$\left[\frac{r^2 - r_0^2 + (z_0 - z)^2}{a-b}\mathrm{E}_\alpha(k) - F_\alpha(k)\right]n_r +$$

$$+\frac{4}{\sqrt{a+b}}\frac{z_0 - z}{a-b}\mathrm{E}_\alpha(k)n_z$$

$$\Phi(P,P_0) = \frac{4}{\sqrt{a+b}}F_\alpha(k) \tag{14}$$

$$E_\alpha(k) = (-1)^\alpha (1 - 16\alpha^2)$$

$$\int_0^{\frac{\pi}{2}} \cos 2\alpha l\psi \sqrt{1 - k^2 \sin^2 \psi}\, d\psi$$

$$F_\alpha(k) = (-1)^\alpha \int_0^{\frac{\pi}{2}} \frac{\cos 2\alpha}{\sqrt{1 - k^2 \sin^2 \psi}}$$

$$a = r^2 + r_0^2 + (z - z_0)^2$$

$$b = 2rr_0$$

$$k^2 = \frac{2b}{a+b}$$

$$l = 1$$

Formulas (13) make it possible to reduce two-dimensional integrals (11) to one-dimensional ones calculated over the generatrix of the composed shell and along radius of the free surface. The singular integral equation obtained as a result of applying formulas (13) to integrals (11) is solved by the discrete singularity method [10].

In formulas (3.5), the parameter α means the number of nodal diameters (or wave number). If $\alpha = 0$, then axisymmetric vibrations are considered.

4. METHODOLOGY

The problem of fluid oscillations in a rigid spherical shell is considered as a test one. Consider a spherical shell of radius $R = 1$ m, partially filled with the ideal incompressible fluid, with filling level h. A numerical analysis was carried out for $(0.2 < h/R < 1.99)$, $h_1 = h/R$. The boundary element method (BEM), described above, and the analytical approach [11] are applied. When using the boundary element method, 150 elements were applied along radius of the free surface and 300 elements along the wetted part of the generatrix. Boundary elements with a constant approximation density are considered, which corresponds to the ideology of the discrete singularities method [10]. A further increase in the number of elements did not lead to a significant change in the results. Table 1 shows the results of calculating the frequencies of axisymmetric fluid vibrations (in Hz) using these methods.

TABLE 1. Frequencies of axisymmetric fluid oscillations in a spherical shell.

Method	**Level of filling h, m**				
	h_1=0.2	h_1=0.6	h_1=1.0	h_1=1.8	h_1=1.99
[11]	3.8261	3.6501	3.7451	6.7641	29.050
BEM	3.8314	3.6510	3.7456	6.7665	29.181

TABLE 2. Frequencies of axisymmetric fluid oscillations in the system of coaxial shells.

R_1/R_2	0.0	0.01	0.2	0.4	0.6	0.9
ω	4.247	4.247	4.086	3.785	3.516	3.212

5. EXPERIMENTAL RESULTS

Two cylindrical coaxial shells of different radii are considered. Let R_1 be radius of the inner shell, R_2 be radius of the outer shell, and H is the level of liquid filling the inner part of the shell structure. The calculations are carried out using 150 boundary elements with the constant approximation of the density along radius of the free surface and 300 elements along the generatrix of each shell. Table 2 shows the vibration frequencies for the compound system of coaxial cylindrical shells. It is assumed that $R_2 = H = 1$ m. Various values of internal radius are considered. Table 2 shows the values of the fluid oscillation frequencies in a system of coaxial shells at $\alpha = 1$ for different R_1/R_2.

The modes of fluid oscillations in the system of coaxial shells are shown in Fig. 2 and Fig. 3.

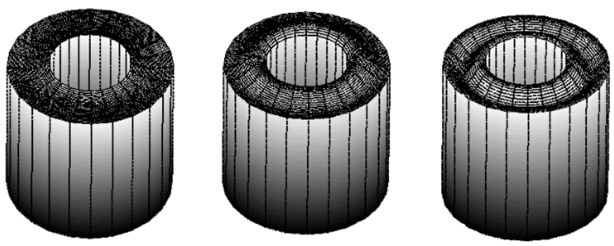

Fig. 2 The first axisymmetric modes of fluid oscillations in the system of coaxial cylindrical shells, $\alpha = 0$.

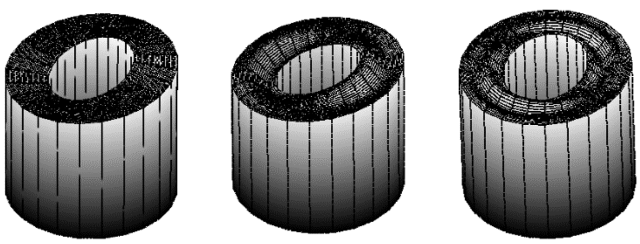

Fig. 3 The first non-axisymmetric modes of fluid oscillations in the system of coaxial cylindrical shells, $\alpha = 1$.

Modes of fluid oscillations shown in Fig. 2 and Fig. 3 correspond to the following ratio R_1/R_2=0.5. Similarly to the modes of liquid vibrations in cylindrical and conical shells considered in [9], we see that the modes of the free liquid surface oscillations have a character inherent in the Bessel functions. The obtained forms oscillations are orthogonal and can be used in solving problems of fluid forced oscillations in coaxial shells, and as a basic system functions in the studying nonlinear oscillations.

6. CONCLUSIONS

The methods of boundary integral equations and discrete singularities are further developed in solving problems of fluid oscillations in rigid coaxial shells, when the free surface the fluid has the ring form. The problem determining the velocity potential and fluid pressure is reduced to solving the system of one-dimensional singular equations. An effective numerical method for its solving has been developed. The algorithm was tested, and the required number of boundary elements is established to obtain the specified accuracy.

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