Near-Flat Limit of Schrodinger$_5$ times S^5

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Abstract. Schrödinger space-time is a manifold, whose holography dual space is the group manifold of the Schrödinger group or in other words the symmetry group of the Schrödinger equation. The analysis of this manifold is of interest from the point of view of holography and the AdS/CFT correspondence. Such geometries, however, pose problems because of their non-trivial structure. Thus, we employ limiting methods, such as the "pp-wave" limit (which is an analogy of the plane-waves from electrodynamics). In our work, we carry out the so-called "near-flat" limiting procedure, which is less restrictive on the geometry in the sense that it preserves more structure form the original manifold. Our goal is to construct a string sigma model, derive the equations of motion and to fix the gauge invariance in a meaningful way.

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1. NEAR-FLAT LIMIT OF SCHRÖDINGER SPACE

We start by considering a general metric on the Cartesian product of 5-dimensional Schrödinger space and a 5-sphere:

$$\frac{ds^2}{l^2} = -\left(1+\frac{\mu^2}{Z^4}-\frac{X_1^2+X_2^2}{Z^2}\right)dT^2 + \frac{1}{Z^2}\left(2dTdV + dX_1^2 + dX_2^2\right) + ds_{S^5}^2,$$

Where the S^5 part is given as a Hopf fiber over S^2,

$$\frac{ds_{S^5}^2}{l^2} = \left(d\chi + P\right)^2 + ds_{CP^2}^2$$

$$ds_{CP^2}^2 = d\eta^2 + \frac{1}{4}\sin^2\eta\left(\sigma_1^2+\sigma_2^2+\cos^2\eta\sigma_3^2\right).$$

This form of the metric is well suited for our work, as the Mauer-Cartan one-forms for it are well known:

$$\sigma_1 = \cos\alpha_2 d\alpha_1 + \sin\alpha_1 \sin\alpha_2 d\alpha_3$$
$$\sigma_2 = \cos\alpha_2 d\alpha_1 - \sin\alpha_1 \cos\alpha_2 d\alpha_3$$
$$\sigma_3 = d\alpha_2 + \cos\alpha_1 d\alpha_3.$$

The expression for P is also known, and is given by

$$P = \frac{1}{2}\sin^2\eta\sigma_3.$$

In order to make a near-flat expansion we must consider an ansatz for the null geodesics on our space. A suitable ansatz for us is to take the following:

$$T = \kappa\tau$$
$$V = \mu^2 m\tau$$
$$X_i = 0$$
$$Z = \sqrt{\frac{\kappa}{m}}$$
$$\eta = 0$$
$$\chi = \omega\tau$$

On the S_5 part we suppose that the null-geodesic is taken along the equator of the 5-sphere. It is trivial to show that this ansatz indeed gives a geodesic curve on our manifold.

Now, in order for our geodesic to be null, we have to impose an additional constraint on the constants:

$$\kappa^2 = \mu^2 m^2 + \omega^2$$

Another useful feature of the Hopf fiber version of the metric is that we know the expression for the B-field,

$$\alpha' B = \frac{l^2\mu}{Z^2} dT \wedge \left(d\chi + P\right).$$

In order to perform a near-flat expansion along the null geodesic, we add the following fluctuations to the ansatz.

$$T = \kappa\left(\sqrt{g}\sigma^+ + \frac{\tau}{\sqrt{g}}\right)$$

$$V = \mu^2 m\sqrt{g\sigma^+} + \frac{v}{m\sqrt{g}}$$

$$X_i = \sqrt{\frac{\kappa}{m}}\frac{X_i}{\sqrt{g}}$$

$$Z = \sqrt{\frac{\kappa}{m}}\left(1 + \frac{Z}{\sqrt{g}}\right)$$

$$\eta = \frac{\eta}{\sqrt{g}}$$

$$\chi = \omega\sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}}$$

This action has a simple geometric interpretation – we are performing a boost along the null geodesic and then allowing for small perturbations along the boosted directions.

2. CONSTRUCTING THE LAGRANGIAN

We have arrived at the expressions for our coordinate fields and now we must construct a dynamical sigma model, or in other words we must substitute our results in the expressions for the metric and the B-field and construct the corresponding Lagrangian. Substituting into the expressions for the metric and the B-field (and taking the limit $g \to \infty$ we get, respectively

$$\alpha' B = \frac{1}{2}\sqrt{g}\,\mu m\eta d\sigma^+ \wedge \sigma_3 + g\,\mu m d\sigma^+ \wedge \left(d\chi - \omega d\tau\right) + \mu m d\tau \wedge d\chi$$

And

$$\begin{aligned} ds^2 = &-\left(\kappa + \mu^2 m^2\right)d\tau^2 \\ &- g\kappa^2\left(X_1^2 + X_2^2\right)d\sigma^{+^2} \\ &- 2\kappa^2\left(g + X_1^2 + X_2^2\right)d\sigma^+ d\tau \\ &+ 2g d\sigma^+ dv + 2g\omega d\sigma^+ d\chi \\ &+ \sqrt{g}\omega\eta^2 d\sigma^+\sigma_3 + 2d\tau dv \\ &+ dX_1^2 + dX_2^2 + dZ^2 + d\chi^2 \\ &+ d\eta^2 + \frac{1}{4}\eta^2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \end{aligned}$$

One can spot that both expressions contain divergent terms, namely the ones containing $\sqrt{g}$. Some of those terms turn out to be total derivatives, but two of them, namely the term

$$\frac{1}{2}\sqrt{g}\,\mu m\eta d\sigma^+ \wedge \sigma_3$$

in the expression for the B-filed and the term

$$\sqrt{g}\omega\eta^2\sigma^+\sigma_3$$

in the expression for the metric remain problematic. We will deal with those terms in the following chapter.

We introduce a new set of Cartesian coordinates,

$$\left(Z, \chi, X_1, X_2\right) \to \left(y_1, y_2, y_3, y_4\right),$$

$$d\eta + \frac{1}{4}\eta^2\sum_{i=1}^{3}\sigma_i^3 = \sum_{i=5}^{8} dy_i^2$$

This enables us to re-write the near-flat background metric and B-field in a more tangible form:

$$\begin{aligned} ds^2 = &-\left(\kappa^2 + \mu^2 m^2\right)d\tau^2 \\ &- g\kappa^2\left(y_3^2 + y_4^2\right)d\sigma^+ - 2\kappa\left(g + y_3^2 + y_4^2\right)d\sigma^+ d\tau \\ &+ 2g d\sigma^+ dv + 2g\omega d\sigma^+ dy_2 \\ &+ 2d\tau dv + \sum_{i=1}^{8} dy_1^2 \\ &+ 2\sqrt{g}\omega\left(y_5 dy_6 - y_6 dy_5\right) \\ &+ y_7 dy_8 - y_8 dy_7 d\sigma^+ \end{aligned}$$

$$\alpha' B = g\mu m d\sigma^{+} \wedge \left(dy_2 - \omega d\tau\right) + \mu m d\tau \wedge dy_2 + \sqrt{g}\,\mu m d\sigma^{+} \wedge \left(y_5 dy_6 - y_6 dy_5 + y_7 dy_8 - y_8 dy_7\right)$$

We have arranged the expressions above in such a way that the divergent terms are the last term in each equation. Furthermore, we've used that

$$\eta^2 \sigma_3 = 2\left(y_5 dy_6 - y_6 dy_5 + y_7 dy_8 - y_8 dy_7\right).$$

Finally, we can construct the Lagrangian for this reduced geometry. In general, the Lagrangian $\Im$ for such theories comprises of fields coupling to the metric and to the B-field,

$$\Im = \sqrt{h}\left(h^{\pm}\partial_{+}X^{\mu}\partial_{-}X^{\nu}G_{\mu\nu}\right) - \varepsilon^{\pm}\partial_{+}X^{\mu}\partial_{-}X^{\nu}B_{\mu\nu}$$

In our notation, this reduces to

$$\Im = -2\partial_{+}X^{\mu}\partial_{-}X^{\nu}G_{\mu\nu} + 2\partial_{+}X^{\mu}\partial_{-}X^{\nu}B_{\mu\nu}$$

We can now insert the expressions for the metric and the B-field and try to figure out a way for the divergent terms to cancel out. It turns out that for that to happen we must impose an additional restriction to our constants, namely

$$\mu m = \omega\alpha'$$

Combining this condition with the previous constraint on the constants, we arrive at the final expression for these parameters:

$$\kappa^2 = \omega^2\left(1 + \alpha'^2\right).$$

After all these manipulations, the Lagranian for the near-flat space limit of our original $Sch_5 \times S^5$ geometry is

$$\Im = 2\omega^2\left(1 + 2\alpha'^2\right)\partial_{+}\tau\partial_{-}\tau + 2\omega^2\left(1 + 2\alpha'^2\right)\left(y_3^2 + y_4^2\right)\partial_{\tau} + 2\left(\partial_{+}\tau\partial_{-}\tau + \partial_{+}\nu\partial_{-}\tau\right) + 2\omega\left(\partial_{+}\tau\partial_{-}y_2 - \partial_{+}y_2\partial_{\tau}\right) - 2\sum_{i=1}^{8}\partial_{+}y_i\partial_{y_i}$$

With this we have performed the near-flat expansion and arrived at the corresponding sigma model.

3. EQUATION OF MOTION AND THE STRESS-ENERGY TENSOR

It is straight forward to obtain the equations of motion for such a theory,

$$2\omega^2\left(1 + 2\alpha'\right)\partial_{+}\tau\partial_{-}\tau - 2\partial_{+}\partial_{-}\nu + \omega^2\left(1 + \alpha'^2\right)\partial_{-}\left(y_3^2 + y_4^2\right) = 0$$

$$\partial_{-}\partial_{+}\tau = 0$$

$$\partial_{+}\partial_{-}y_i + \omega^2\left(1 + \alpha'^2\right)y_i\partial_{\tau} = 0, \qquad i = 3,4$$

$$\partial_{+}\partial_{-}y_k = 0, \qquad k = 1,2,5,6,7,8.$$

From here we can arrive at the expressions for some chiral currents

$$j_{+}^{\tau} = 2\omega^2\left(1 + 2\alpha'^2\right)\partial_{+}\tau - 2\partial_{+}\nu + \omega^2\left(1 + 2\alpha'^2\right)\left(y_3^2 + y_4^2\right),$$

$$j_{+}^{\nu} = \partial_{+}\tau$$

$$j_{+}^{y_k} = 2\omega\partial_{+}y_k \qquad k = 1,2,5,6,7,8$$

It is important to note that these might not be *real* conserved currents – they might not correspond to a continuous symmetry of the theory.

The components of the energy-momentum tensor are given, as usual, by the expression

$$T_{++} = G_{\mu\nu}\partial_{+}X^{\mu}\partial_{+}X^{\nu}$$

$$T_{--} = G_{\mu\nu}\partial_{-}X^{\mu}\partial_{-}X^{\nu}$$

In our case this translates to a fairly long expression. However, this expression is linear in most terms:

$$T_{++} = -2\omega^2\left(1 + \alpha'^2\right)\partial_{+}\tau + 2\partial_{+}\nu - \omega^2\left(1 + \alpha'^2\right)\left(y_3^2 + y_4^2\right) + 2\omega\partial_{+}y_2$$

$$T_{--} = \frac{1}{g}\left(-\omega^2\left(1 + 2\alpha'^2\right)\partial_{-}\tau\partial_{-}\tau + 2\partial_{-}\tau\partial_{-}\nu + \sum_{i=1}^{8}\partial_{-}y_i\partial_{-}y_i\right)$$

4. CONCLUSION

It is straightforward to obtain the equations of motion for such a theory We've investigated the near-flat limit of our chosen geometry and have constructed a divergent-free sigma model near a null geodesic curve. The equations of motion as well as the components of the stress-energy tensor have been derived and investigated. As further work, we must perform a gauge fixing, investigate further the effective theory and investigate the holographic duality present due to the fact that the geometry comes from a TsT transformation. We must compare this non-relativistic result to known relativistic ones and examine how this affects the holographic dictionary.

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