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On the geodesics in Bondi-Gold-Hoyle universe model

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Abstract. In this paper we first derive the geodesic equations in the Bondi-Gold-Hoyle universe model and then we provide exact analytical solutions of these equations. The obtained solutions describe the motion of the massless particles in the considered universe model.

Keywords: Universe models, Steady-State theory, differential geometry, differential equations, geodesics equations.

1. INTRODUCTION

Let $p=(u^1,u^2,u^3,u^4)$ be an arbitrary point in a Minkowski space M^4. We define in M^4 a nondegenerate metric tensor g by

$$g_{ij}=g\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right),\quad i,j=1,\dots,4$$

and a curve $\gamma\in M^4$ by

$$\gamma: u^k=u^k(s),\quad k=1,\dots,4.$$

It is well known (see, *e.g.*, Weinberg, 1972 [p. 71]; Busemann, 2005; Toponogov, 2006 [p. 156] and references therein) that γ is called *geodesic* if it satisfies the following equations

$$\frac{d^2u^k}{ds^2}+\Gamma^k_{ij}\frac{du^i}{ds}\frac{du^j}{ds}=0, \tag{1}$$

where Γ^k_{ij} are the Christoffel symbols defined by

$$\Gamma^k_{ij}=\frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial u^i}+\frac{\partial g_{il}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^l}\right), \tag{2}$$

where (g^{kl}) is the inverse of the matrix (g_{kl}).

Let p be a point on the curve $\gamma\in M^4$. Define the tangent vector in the point p by

$$\dot{\gamma}_p=\left(\frac{du^1}{ds},\frac{du^2}{ds},\frac{du^3}{ds},\frac{du^4}{ds}\right).$$

Then, we have

$$\dot{\gamma}_p^2=g_{ij}\frac{du^i}{ds}\frac{du^j}{ds} \tag{3}$$

in terms of Einstein summation convention. Recall that, if $\dot{\gamma}_p$ is a nonzero vector, then the curve $\gamma\in M^4$ is called *timelike* in the case $\dot{\gamma}_p^2<0$, *lightlike* or *null* in the case $\dot{\gamma}_p^2=0$ and *spacelike* in the case $\dot{\gamma}_p^2>0$ (see Yilmaz & Turgut, 2008; İlarslan & Nešović, 2009 and references therein).

In terms of general relativity, the geodesics represent the paths of freely moving particles in a curved spacetime (see, *e.g.*, Grøn & Næss; 2002 [Sect. 12.8]). Therefore, it is not surprising that one of the most interesting problems in mathematical physics is solving the geodesic equations (1). Note that, in Cartesian coordinates, we have $\Gamma^k_{ij}=0$ (for all i,j) and the equations (1) become equations of straight lines.

It can be seen by (1), that the geodesic equations depend on the metric ds^2, *i.e.*, on the geometry of the considered model. Depending on the metric, there are a lot of models of the Universe, presented in the literature (see, *e.g.*, Carroll, 2004, Hawley & Holcomb, 2005, Weinberg, 2008 and references therein). Nevertheless, cosmology is dominated by two alternative paradigms. The first one is called *Big Bang model* and the second one is called

Steady-State theory (see Aguirre & Gratton, 2002). The Steady-State theory was first presented by Einstein in 1931 (in unpublished manuscript). Seventeen years later, steady-state models of the expanding universe were independently proposed by, Bondi & Gold, 1948 and Hoyle, 1948. In this paper, we refer the Steady-State theory as *Bondi-Gold-Hoyle universe model*. For more detailed historical survey of this model, we refer the reader to O'Raifeartaigh & Mitton, 2015.

The main aim of this paper is twofold: first to establish the geodesic equations in the Bondi-Gold-Hoyle universe model and second to obtain analytical solutions of these equations in the case of lightlike geodesics.

2. THE GEODESIC EQUATIONS

Let $u=(t,\vec{u})\in M^4$, then the Bondi-Gold-Hoyle model can be defined by the following de Sitter sphere (see Spradlin et al., 2001):

$$S_1^4:\left|\begin{array}{l} z^1 = r\sinh\varphi + \dfrac{e^{\varphi}}{2r}|\vec{u}|^2; \\ z^2 = r\cosh\varphi - \dfrac{e^{\varphi}}{2r}|\vec{u}|^2; \\ z^3 = e^{\varphi}u^2;\ z^4 = e^{\varphi}u^3;\ z^5 = e^{\varphi}u^4, \end{array}\right. \tag{4}$$

where r is the radii of S_1^4, $\varphi = t/r$ and $|\vec{u}|^2 = (u^2)^2 + (u^3)^2 + (u^4)^2$. From (4), we get the following metric tensor

$$g_{ij} = g\left(\frac{\partial\vec{z}}{\partial u^i},\frac{\partial\vec{z}}{\partial u^j}\right) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{2\varphi} & 0 & 0 \\ 0 & 0 & e^{2\varphi} & 0 \\ 0 & 0 & 0 & e^{2\varphi} \end{pmatrix} \tag{5}$$

Combining equations (5) and (2), we obtain the following nonzero Christoffel symbols

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{e^{2\varphi}}{r},$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \Gamma^4_{14} = \Gamma^4_{41} = \frac{1}{r}.$$

Hence, taking into account (1) for the geodesic equations of the Bondi-Gold-Hoyle universe model, we get the system

$$\left|\begin{array}{l} \dfrac{d^2t}{ds^2} + \dfrac{e^{2\varphi}}{r}\left(\left(\dfrac{du^2}{ds}\right)^2 + \left(\dfrac{du^3}{ds}\right)^2 + \left(\dfrac{du^4}{ds}\right)^2\right) = 0 \\ \dfrac{d^2u^2}{ds^2} + \dfrac{2}{r}\dfrac{dt}{ds}\dfrac{du^2}{ds} = 0 \\ \dfrac{d^2u^3}{ds^2} + \dfrac{2}{r}\dfrac{dt}{ds}\dfrac{du^3}{ds} = 0 \\ \dfrac{d^2u^4}{ds^2} + \dfrac{2}{r}\dfrac{dt}{ds}\dfrac{du^4}{ds} = 0. \end{array}\right. \tag{6}$$

On the other hand, from (3) and (5), in the case of lightlike curve, we get

$$e^{2\varphi}\left(\left(\frac{du^2}{ds}\right)^2 + \left(\frac{du^3}{ds}\right)^2 + \left(\frac{du^4}{ds}\right)^2\right) = \left(\frac{du^1}{ds}\right)^2$$

Taking into account the last identity with $u^1 = t$, we obtain the following solutions of the geodesic equations (6)

$$\begin{array}{l} t = r\ln\left|\dfrac{c_1 s + c_2}{r}\right|, \\ u^2 = c_3 - \dfrac{r^2}{c_1 s + c_2}, \\ u^3 = c_4 - \dfrac{r^2}{c_1 s + c_2}, \\ u^4 = c_5 - \dfrac{r^2}{c_1 s + c_2}, \end{array} \tag{7}$$

where $c_i \in \mathbb{R}$ $(i = 1,\ldots,5)$ and $c_1 s + c_2 \neq 0$. One can see from the first equality in (7) that $r = c_1 s + c_2$ implies $t = 0$. This means that for the massless particles in the Bondi-Gold-Hoyle universe model there exist infinitely many initial moments of time.

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