Podolsky Generalized Electrodynamics Lower Bounds on the Mass of the Dark Photon

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Abstract. In the current paper we present the basics of generalized electrodynamics. One of the key features of this theory is that the solution can be represented as the motion of two particles: the massless photon from standard Maxwell electrodynamics and new massive particle, called dark photon, which is a candidate for particle of the dark matter. In the current research we use two quantum mechanical approaches, perturbation theory and variational method in order to find the energy contribution to the ground state of the hydrogen atom coming from the generalized electrodynamics. We will compare our results with the experimental data in order to find a lower bound for the mass of the dark photon. Our bound, calculated using perturbation theory is 52 MeV, while the variational method produces much higher result of 28 GeV.

Keywords: Dark photons, generalized electrodynamics, perturbation theory

1. INTRODUCTION

Classical Maxwell electrodynamics describes quite accurately all observed electromagnetic phenomena. However, the theory encounters difficulties when dealing with point charges since it predicts infinite energies. There are various ways to deal with this problem. The most popular one is quantum electrodynamics, which uses renormalization techniques. Other ways are the generalised theories such as Born-Infeld non-linear electrodynamics (Born & Infeld, 1934) and Podolsky generalised electrodynamics (Podolsky & Schwed, 1948). The latter is a matter of interest in the recent years since it gives possible candidates for dark matter particles.

In the current paper we take advantage of two quantum-mechanical approaches, namely perturbation theory and energy variational method, in order to estimate the lower bound for the mass of the dark photon. The later can be achieved by calculating the energy of the ground state of hydrogen atom by the above-mentioned methods and comparing the results with the existing experimental data. Our calculations of the lower bound using perturbation theory shows at least 52 MeV for the mass of the dark photon, while the variational method, using dimensionless variable, produces much higher result of 28 GeV.

2. BASICS OF PODOLSKY THEORY

Generalized electrodynamics, also known as Podolsky electrodynamics, is defined by the following Lagrangian density (Gratus et al., 2015):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{l^2}{4} \partial^\alpha F_{\mu\nu} \partial_\alpha F^{\mu\nu} - j_\mu A^\mu, \qquad (1)$$

Where j_μ is the four-current and $F_{\mu\nu}$ is the curvature tensor, defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (2)$$

Here $A_\mu = (\phi, \vec{A})$ is the gauge electromagnetic potential in natural units, $\hbar = c = 1$. The Einstein's summation convention is also implied.

The first term in Eq. (1) is the Lagrangian density of the Maxwell theory. The last term represents the sources of the field. The second term was first introduced by F. Bopp in 1940

(Bopp, 1940) and independently by B. Podolsky in 1942. Here the parameter l is a new fundamental constant with dimension of length.

Since the standard electrodynamics has been experimentally proven with high precision the contribution of this term and therefore l should be small.

One can derive the equation of motion by the principle of least action:

$$S = \int \mathcal{L} d^4 x, \qquad \delta S = 0. \tag{3}$$

After varying with respect to the gauge field A_μ, we find the following fourth order differential equation (in Lorentz gauge):

$$\Box\left(1 - l^2 \Box\right) A^\mu = -j^\mu, \quad \partial_\mu A^\mu = 0. \tag{4}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alambert's operator.

Since second order partial differential equations are more common in physics, the methods for finding their solutions are well-developed. In order to reduce our problem to such equations we will introduce the transformation (Gratus et al., 2015):

$$\hat{A}_\mu = A_\mu - l^2 \Box A_\mu, \quad \tilde{A}_\mu = -l^2 \Box A_\mu. \tag{5}$$

It is obvious that

$$A_\mu = \hat{A}_\mu - \tilde{A}_\mu, \tag{6}$$

thus the corresponding second order equations are

$$\Box \hat{A}_\mu = -j_\mu, \tag{7}$$

$$\Box \tilde{A}_\mu - l^{-2} \tilde{A}_\mu = -j_\mu. \tag{8}$$

First of these equations gives the standard massless photon from the Maxwell theory. The second one describes massive particle with Compton wavelength $l \sim 1/m_{pod}$. This particle is called the Podolsky dark photon.

Eqs. (7) and (8) can be solved in the electrostatic case using Green functions and Fourier transform (Lande and Thomas, 1941). This gives the electrostatic potential in the form:

$$\phi = \frac{q}{r}\left(1 - e^{-\frac{r}{l}}\right). \tag{9}$$

From this expression we can verify that in the limit, $r \to 0$, one finds $\phi \to q/l$, which is finite. The latter means that the energy of a point charge in the whole space is also finite, which is the main motivation for considering Podolsky electrodynamics.

3. LOWER BOUNDS FOR THE MASS OF THE DARK PHOTON

In order to find a maximum value for the parameter l, which means a minimum value for the mass of the dark photon, we will consider solutions for the ground state of the hydrogen atom with Podolsky's potential and then compare the result to the experimental data. Since in this case the Schrödinger equation cannot be solved exactly we will use two approximate methods – perturbation theory and Ritz-Hileras variational method.

3.1 Perturbation theory

First, we are going to find the ground state of the hydrogen atom in the generalized electrodynamics using perturbation theory. The problem is essentially reduced to solving the time independent Schrödinger equation:

$$\hat{H} \Psi_n = E_n \Psi_n \tag{10}$$

with the Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{V}. \tag{11}$$

Here the unperturbed problem is given by

$$\hat{H}_0 = \frac{\hat{p}^2}{2\mu} - \frac{q^2}{r}, \tag{12}$$

and the small perturbation potential is the Podolsky potential:

$$\hat{V} = \frac{q^2}{r} e^{-r/l}. \qquad (13)$$

The solution to Eq. (10) with $\hat{H}_0$ is well-known result for the Hydrogen from the non-relativistic quantum mechanics. It gives the energy in the ground state of the Hydrogen atom:

$$E_0 = -\frac{\mu q^2}{2\hbar^2} = -13,598287\ eV. \qquad (14)$$

Now, we are interested in the contribution to this energy coming from the perturbation potential (13). One can calculate the first order correction of the energy of the ground state, which, in this case, is given by

$$E_1 = \langle \Psi_0 | \hat{V} | \Psi_0 \rangle = \frac{16\pi}{r_0} \frac{q^2}{(2 + r_0\beta)^2}, \qquad (15)$$

where, for convenience, we have introduced the parameter $\beta = 1/l$. We expect that Eq. (15) leads to a small contribution. From the table bellow one finds that the best agreement with the experimental data,

$$E_{exp} = -13,598434\ eV, \qquad (16)$$

is achieved by $\beta = 4,2 \times 10^{13}\ m^{-1}$, or $l = 1/\beta$ $= 2,38 \times 10^{-14}\ m$.

TABLE 1. Some numerical values of the Podolsky parameter β and their corresponding ground state energies in the perturbation theory.

β, m^{-1}	**$E_{0,numerical}^{perturbation}$, eV**
$5,0 \times 10^{10}$	+49,697
$5,0 \times 10^{11}$	-11,912
$3,0 \times 10^{13}$	-13,597
$4,2 \times 10^{13}$	-13,598
$5,0 \times 10^{14}$	-13,598
$5,0 \times 10^{15}$	-13,598

Bigger values for β contribute very little after the third digit in the energy of the ground state and thus no further digits can agree with (16). Using the Compton's wave length formula one finds the lower bound on the mass of the dark photon in Podolsky theory[1]:

$$m_\gamma^{lower} = \frac{2\pi\hbar}{lc} \approx 52\, MeV \qquad (17)$$

As of now all experiments looking for massive photons show negative results in the MeV band. This results don't necessarily exclude the existence of a dark photon with mass of 52 MeV or heavier. It is possible that the existing experiments aren't enough sensitive to detect such a particle.

3.2 Variational method

In this subsection we will find a lower bound for the mass of the dark photon using Ritz-Hileras variational method. We will consider the energy functional given by

$$E = \langle \hat{H} \rangle = \langle \Psi | \hat{H} | \Psi \rangle, \qquad (18)$$

where $\Psi = \Psi(a,b,...)$ is some appropriate probe wave function and $a,b,...$, are some parameters to be calculated. The probe wave function should be normalised,

$$\langle \Psi | \Psi \rangle = 1. \qquad (19)$$

It should also be continuous and quadratic integrable. Boundary conditions, symmetries and other physical properties should be taken also into account. Since the ground state is defined as the state with minimal energy, the following equations must be valid:

[1] The values of the fundamental physical constants and data were taken from NIST (National Institute of Standards and technology).

$$\frac{\partial E(a,b,...)}{\partial a}=0, \quad \frac{\partial E(a,b,...)}{\partial b}=0, \quad ... \qquad (20)$$

Therefore, we have *n* equations for *n* parameters. An appropriate function is given by

$$\Psi = Ae^{-\alpha r}. \qquad (21)$$

From condition (19) one finds $A=\sqrt{\alpha^3/\pi}$. Therefore the energy functional (18) is given explicitly by

$$E(\alpha)=\frac{\hbar^2\alpha^2}{2\mu}-q^2\alpha+\frac{4\alpha^3q^2}{(2\alpha+\beta)^2}. \qquad (22)$$

Here, we will consider the energy functional to depend only on the parameter *α*, and the Podolsky parameter *β* will be related to *α* via Eqs. (20). Thus, the conditions from (20) forces us to find the first derivative $dE(\alpha)/d\alpha=0$, which leads to a relation between *α* and *β*, in this case a fourth degree algebraic equation,

$$\alpha^4+\frac{3\beta}{2}\alpha^3+\frac{3\beta^2}{4}\alpha^2 + \frac{\beta^2(\hbar^2\beta-6q^2\mu)}{8\hbar^2}\alpha-q^2\frac{\mu\beta^3}{8\hbar^2}=0. \qquad (23)$$

This result was obtained by Cuzinatto, De Melo, Medeiros and Pompeia(Cuzinatto et al., 2011). In the cited paper the authors have considered only the terms up to first order in *l* (They have neglected the first two terms in the above equation). Which leads to 35.51 MeV for the lower dark photon mass.

Now we are going to examine the full analytical solutions by taking advantage of the numerical and the symbolic capabilities of the software package Wolfram Mathematica to deal with the roots of the given equation. We expect that for some large β one should get the known value for α, which for the Hydrogen problem in the ground state is (the inverse of the Bohr radius):

$$\alpha = 1,888697483\times 10^{10}\ m^{-1}. \qquad (24)$$

After some numerical experiments with β, one finds that only one of the four roots gives real and positive values for α. Now, after we have identified which root (it has lengthy form to wright it here) we can plug it in the energy functional (22) to obtain the best agreement with the experimental data (16). In the table below, we see that the values of the energy converge to the experimental one from (16) up to three valid digits after the coma for increasing values of *β*.

TABLE 2. Some numerical values of the Podolsky parameter *β* and their corresponding ground state energies in the Ritz-Hileras variational method.

β, m^{-1}	**$E_{0,numerical}^{variational}$, eV**
$5,0\times10^{10}$	-10,101
$5,0\times10^{11}$	-13,466
$5,0\times10^{12}$	-13,596
$1,2\times10^{13}$	-13,598
$5,0\times10^{14}$	-13,598
$5,0\times10^{15}$	-13,598

The minimal value of *β*, still giving visible agreement with Eq. (16), is achieved by

$$\beta = 1,2\times 10^{13}\ m^{-1}. \qquad (25)$$

Thus $l=1,833\times10^{-14}\ m$, and the lower bound on the mass of the dark photon in this case is

$$m_\gamma^{lower}=14,9\ MeV. \qquad (26)$$

As one can note this bound has the same order of magnitude as the one found by perturbation theory in Eq. (17).

A better idea is to use dimensionless variables,

$$a=\alpha r_0, \quad b=\beta r_0, \quad r_0=\frac{\hbar^2}{\mu q^2}, \qquad (27)$$

Where

$$r_0 = \frac{1}{\alpha} = 5.2917721067 \times 10^{-11} \qquad (28)$$

is the Bohr radius. In this case Eq. (23) can be rewritten in the following form

$$a^4 + \frac{3b}{2}a^3 + \frac{3b^2}{4}a^2 + \frac{b^2(b-6)}{8}a - \frac{b^3}{8} = 0. \qquad (29)$$

Since now $a = 1$ for the unperturbed ground state, we look for such b that gives a as close to 1 as possible, which is equivalent to maching the numerical and the experimental results for α (which is the inverse of the Borh's radius). With some numerical play, shown in the table below,

TABLE 3. Some numerical values of the dimensionless Podolsky parameter b and the corresponding values of the Borh's radius in the Ritz-Hileras variational method.

b	$r_0^{numeric}$**, m**
$1,0\times10^{1}$	$4,9400255011\times10^{-11}$
$1,0\times10^{2}$	$5,2857616677\times10^{-11}$
$1,0\times10^{3}$	$5,2917089443\times10^{-11}$
$1,0\times10^{5}$	$5,2917721004\times10^{-11}$
1.2×10^{6}	$5,2917721067\times10^{-11}$
$1,0\times10^{8}$	$5,2917721067\times10^{-11}$

one obtains the minimum value $b = 1,2\times10^6$ to agree best with the Borh's radius from Eq. (28) up to all valid digits after the comma. Thus the lower bound on the mass of the dark photon in this case

$$m_\gamma^{lower} = 28\,GeV. \qquad (30)$$

This value is two orders of magnitude bigger than the previously obtained values for the lower bound of the mass of the dark photon in the Podolsky generalized electrodynamics. The latter is due to the numerical calculations with dimensionless equations and the comparison of the numerical value to the experimental value of the Borh's radius instead of the energy of the hydrogen in the ground state. However, this value on the lower mass of the dark photon is closer to the expected one as given in (Accioly & Scatena, 2010; Buffalo et al., 2014), due to very accurate calculations, involving the magnetic moment of the electron. In general, one expects that the realistic lower bound on the mass of this dark particle to be at least as low as the mass of the W and Z bozons.

4. CONCLUSIONS

The generalized electrodynamics gives a candidate for dark matter particle, called the dark photon. In the current research we have calculated several lower bounds on the mass of this particle through the energy contribution to the ground state of the hydrogen atom, coming from this theory. We have used two methods: perturbation theory, which gives lower bound of 52 MeV, and variational method, for which this bound is 28 GeV. The latter value is as big as the more realistic estimations (~ 42 GeV) of the lower bound on the mass of the Podolsky dark photon, which include relativistic corrections from quantum field theory (Accioly & Scatena, 2010; Buffalo et al., 2014).

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