Degenerate Four-Wave Mixing Between Three Optical Waves, Propagating in Isotropic Nonlinear Medium

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Abstract. In the present work it is investigated the process of degenerate four-wave mixing between three optical waves, described by a system of nonlinear differential equations whose solutions are in the form of Jacobi's elliptic functions.
The main idea of this paper is to investigate the influence of various parameters, participating in the analytical solutions, on the evolution of the three optical waves. Numerical calculations are made. They represent graphically the changes in the waves' intensities and the periodical energy exchange between them. Particular attention has been paid to the dependence of the energy exchange on the initial phase mismatch between the three waves.

Keywords: degenerate four-wave mixing, Jacobi functions, phase mismatch, energy exchange.

1. INTRODUCTION

In the last few decades, the nonlinear phenomena, observed during various experiments with intensive laser light, propagating in isotropic media, are actively investigated by the scientists. Effects like stimulated Raman scattering (SRS), stimulated Brillouin scattering (SBS), four-wave mixing (FWM) and high harmonic generation (HHG) are part of them. For SRS and SBS the medium plays an active role. These processes depend on the media's molecular and density variations (Agrawal, 2007). For the parametric processes like FWM and HHG the medium play a passive role. They generate light in ultraviolet and visible regions of the spectrum and mediate the interaction between different optical waves. FWM and HHG are result of the modulation of the refractive index of medium and require the phase-matching condition to be satisfied so they can be observed. It is important to mention that nonlinear effects like FWM and HHG are related to the second-order susceptibility $\chi^{(2)}$ and third-order susceptibility $\chi^{(3)}$. The third-order parametric processes, like FWM and third-harmonic generation (Boyd, 2003; Shen, 1984; Butcher & Cotter, 1990; Schubert & Wilhelmi, 1986; Armstrong et al, 1962; Hill et al, 1979; Stolen et al, 1981; Stolen et al, 1974; de Araujo et al, 1996; Dmitriev & Tarasov, 1982; Washio et al, 1980) can be easily obtained in isotropic media such as optical waveguides and other nonlinear media.

In (Stolen & Bjorkholm, 1982) authors investigated the parametric four-photon amplification and the generation of new frequencies (signal and idler waves) for a constant pump wave, i.e. without including the change of intensity and the phase of the pump. The expressions for the gain and amplification of the laser pulses, evolving in long optical waveguides, were obtained by using coupled wave equations, modified by overlap integrals, which are similar to the plane wave equations (Stolen & Bjorkholm, 1982). Later, the same effect, without including SRS, was studied in approximation for a given intensity.

The more general task which includes a change in intensity and the phase of the pump wave has been studied in details in (Andreeva et al). The process of FWM between three optical waves, propagating in isotropic nonlinear Kerr-type medium was investigated analytically in the frames of a system of three nonlinear differential equations. Exact analytical solutions which describe the periodic energy exchange between signal, idler and pump waves were found. The solutions were in the form of Jacobi's elliptic sine functions. The mathematical method was applied for waves with arbitrary initial intensities, generalized phase and presence of wave number mismatch.

The main idea of the present work is to present numerical calculations, based on the obtained analytical results in (Andreeva et al), that describes the periodic energy exchange between the three waves for different initial conditions and different initial generalized phase.

2. THEORETICAL BASIS

We are interested in the evolution of three one-dimensional optical pulses, propagating in isotropic nonlinear medium. The pulses interact through the process of degenerate FWM (Fig. 1).

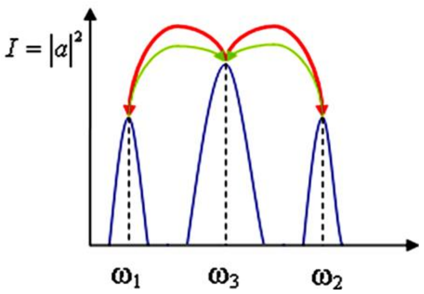

Fig. 1 Images of the process of degenerate four-wave mixing. The expression for the frequency conversion of the three waves is $2\omega_3 = \omega_1 + \omega_2$.

The system of equations, describing the interaction is of the kind:

$$
\begin{aligned}
i\frac{\partial A_1}{\partial z} &= \gamma_1 A_3^2 A_2^* e^{i\Delta kz} \\
i\frac{\partial A_2}{\partial z} &= \gamma_2 A_3^2 A_1^* e^{i\Delta kz} \qquad , \\
i\frac{\partial A_3}{\partial z} &= 2\gamma_3 A_1 A_2 A_3^* e^{-i\Delta kz}
\end{aligned}
\tag{1}
$$

where $A_1(z)$, $A_2(z)$ and $A_3(z)$ are the complex amplitude functions of the signal, idler and pump waves, γ_1, γ_2, γ_3 and Δk are the nonlinear coefficients of the medium and the wave number mismatch $(2k_3 = k_1 + k_2 + \Delta k)$.

The system was analytically solved in (Andreeva et al). Here, we will show its solutions briefly in order to present more precisely the numerical results. To find a solution of the system (1), the following substitutions are made:

$$
\begin{aligned}
A_1 &= a_1(z)\exp(i\varphi_1(z)) \\
A_2 &= a_2(z)\exp(i\varphi_2(z)) \\
A_3 &= a_3(z)\exp(i\varphi_3(z))
\end{aligned}
$$

where $a_1(z) = |A_1(z)|$, $a_2(z) = |A_2(z)|$, $a_3(z) = |A_3(z)|$, $\varphi_1(z)$, $\varphi_2(z)$ and $\varphi_3(z)$ are real functions, describing the amplitude functions and phases of the waves.

Thus, after a couple of transformations, the system of equations (1) takes the form:

$$
\begin{aligned}
i\frac{\partial a_1}{\partial z} - a_1\frac{\partial \varphi_1}{\partial z} &= \gamma_1 a_3^2 a_2 \exp(i\Psi), \\
i\frac{\partial a_2}{\partial z} - a_2\frac{\partial \varphi_2}{\partial z} &= \gamma_2 a_3^2 a_1 \exp(i\Psi), \\
i\frac{\partial a_3}{\partial z} - a_3\frac{\partial \varphi_3}{\partial z} &= 2\gamma_3 a_1 a_2 a_3 \exp(-i\Psi),
\end{aligned}
\tag{2}
$$

where $\psi = 2\varphi_3 - \varphi_2 - \varphi_1 + \Delta kz$ is the generalized phase.

As a next step, we equalize the Real (Re) and Imaginary (Im) parts on both sides of the equations:

$$
\mathrm{Re:}\quad
\begin{aligned}
a_1\frac{\partial \varphi_1}{\partial z} &= -\gamma_1 a_3^2 a_2 \cos\Psi, \\
a_2\frac{\partial \varphi_2}{\partial z} &= -\gamma_2 a_3^2 a_1 \cos\Psi, \\
a_3\frac{\partial \varphi_3}{\partial z} &= -2\gamma_3 a_1 a_2 a_3 \cos\Psi,
\end{aligned}
\tag{3}
$$

$$
\mathrm{Im:}\quad
\begin{aligned}
\frac{\partial a_1}{\partial z} &= \gamma_1 a_3^2 a_2 \sin\Psi, \\
\frac{\partial a_2}{\partial z} &= \gamma_2 a_3^2 a_1 \sin\Psi, \\
\frac{\partial a_3}{\partial z} &= -2\gamma_3 a_1 a_2 a_3 \sin\Psi.
\end{aligned}
\tag{4}
$$

By using the system of equations (4), the following conservation laws were obtained:

$$\gamma_2\gamma_3 a_1^2 + \gamma_1\gamma_3 a_2^2 + \gamma_1\gamma_2 a_3^2 = c = const, \quad (5)$$

$$\gamma_2\gamma_3 a_1^2 - \gamma_1\gamma_3 a_2^2 = c_1 = const, \quad (6)$$

where a_1^2, a_2^2 and a_3^2 are the intensities of the three waves. The equations (5) and (6) present respectively the conservation law for the weighted sum and subtract of the intensities of the waves.

Having in mind the expression for the generalized phase Ψ and the system of equations (3), the following differential equation was found:

$$\frac{d\cos\psi}{dp} = \frac{\left(p(c-p) - (c-p)^2 + \gamma_1 c_1^2\right)}{(c-p)^2 - \gamma_3^2 c_1^2}\cos\Psi + \frac{\sqrt{\gamma_1\gamma_2}\Delta k}{2p\sqrt{(c-p)^2 - \gamma_3^2 c_1^2}} \quad (7)$$

where $p = \gamma_1\gamma_2 a_3^2$. (8)

Its solution is of the kind:

$$p\left[\sqrt{(c-p)^2 - \gamma_3^2 c_1^2}\cos\psi - \frac{\Delta k\sqrt{\gamma_1\gamma_2}}{2}\right] = B_0 = const \quad (9)$$

We search for an expression for $p(z)$. For that reason we use equation (8):

$$\frac{dp}{dz} = -\frac{2}{\sqrt{\gamma_1\gamma_2}}\sqrt{p^2(c-p)^2 - \alpha p^2 - \eta(\beta+p)^2} \quad (10)$$

where

$$\alpha = \gamma_3^2 c_1^2 = const > 0,$$
$$\eta = \frac{\gamma_1\gamma_2\Delta k^2}{4} = const > 0, \quad (11)$$
$$\beta = \frac{2B_0}{\sqrt{\gamma_1\gamma_2\Delta k}} = const > 0.$$

We make another substitution:

$$p = \frac{c}{2} - ky \quad (12)$$

After a couple of transformations, equation (10) can be presented as follow:

$$y'' + \left(1+k^2\right)y - 2k^2y^3 = 0 \quad (13)$$

where $$1 + k^2 = \frac{c}{2} + \alpha + \eta \quad (14)$$

$$\frac{\alpha c + \eta(2\beta + c)}{2k} = 0 \quad (15)$$

The solution of equation (13) is in the form of Jacobi elliptic sine function (Schwalm, 2014):

$$a_1^2 = \frac{1}{2\gamma_2\gamma_3}\left[\frac{c}{2} + k\ sn\left(\frac{2z}{\sqrt{\gamma_1\gamma_2}};k\right) + c_1\right],$$
$$a_2^2 = \frac{1}{2\gamma_1\gamma_3}\left[\frac{c}{2} + k\ sn\left(\frac{2z}{\sqrt{\gamma_1\gamma_2}};k\right) - c_1\right], \quad (17)$$
$$a_3^2 = \frac{1}{\gamma_1\gamma_2}\left[\frac{c}{2} - k\ sn\left(\frac{2z}{\sqrt{\gamma_1\gamma_2}};k\right)\right]$$

where a_1^2, a_2^2 and a_3^2 correspond respectively to the signal, idler and pump waves. It is important to mention that the solutions (17) for the intensities of the three waves are obtained under the conditions (14), (15) and $k \neq 0$:

$$k^2 = 2\left(\frac{c}{2}\right)^2 - 1$$
$$-\frac{\Delta k\sqrt{\gamma_1\gamma_2}}{2}\sqrt{\left(\frac{c}{2}\right)^2 - c_1^2}\cos\psi(0) + \frac{\Delta k\gamma_1\gamma_2}{4} \quad (18)$$
$$1 \geq k^2 \geq 0.$$

3. NUMERICAL CALCULATIONS

We can assume that the solutions (17) are defined by the values of the constants c, c_1, Δk which depend respectively on the initial conditions and the initial generalized phase $\psi(0)$.

When the initial wave number mismatch is set to zero $(\Delta k = 0)$ the process of FWM is the most intensive and:

$$k^2 = \frac{c^2}{2} - 1 \tag{19}$$

As it was mentioned before, it is needed the condition $1 > k_2 > 0$ to be satisfied, so that the obtained expressions (17) to be solutions of the system of equations (1). As a result of that, it is necessary the constant c to be in the frames of $2 > c > \sqrt{2}$, $\sqrt{2} = 1{,}4142$.

When $\Delta k \neq 0$ and $c_1 = 0$, i.e. $\gamma_2 a_1^2 = \gamma_1 a_2^2$, k_2 can be presented as follow:

$$k^2 = 2\left(\frac{c}{2}\right)^2 - 1 - \frac{\Delta k \sqrt{\gamma_1 \gamma_2}}{2} \frac{c}{2} \cos\psi(0) + \frac{\Delta k^2 \gamma_1 \gamma_2}{4} \tag{20}$$

It is clearly seen that in this case the value of k_2 strongly depends, not only on Δk, but also on the initial generalized phase $\psi(0)$.

We did a number of numerical simulations for different initial conditions, which represent the process of energy exchange between the three optical waves through the parametric process of FWM. Since the values of the nonlinear coefficients for the three waves are close, we assume that $\gamma_1 = \gamma_2 = \gamma_3 \approx 1$.

First, we will consider the case when $\Delta k = 0$. From (19), it is clearly seen that Δk doesn't depend on the initial generalized phase $\psi(0)$. It depends only on the sum c of the initial intensities of the three waves. The periodic energy exchange between signal, idler and pump waves for the different values of the constant c is given in the figures below (Fig. 2, Fig. 3 and Fig. 4).

With red line it is presented the intensity of the pump wave, blue and green lines correspond respectively to the signal and idler waves. The intensive energy exchange is clearly seen with the growth of the sum of the initial intensities of the three waves c.

The change in the form of the waves is a result of the properties of the Jacobi's elliptic function sine. The graphic of the function becomes nearly rectangular with the growth of the parameter k, respectively the growth of the constant c.

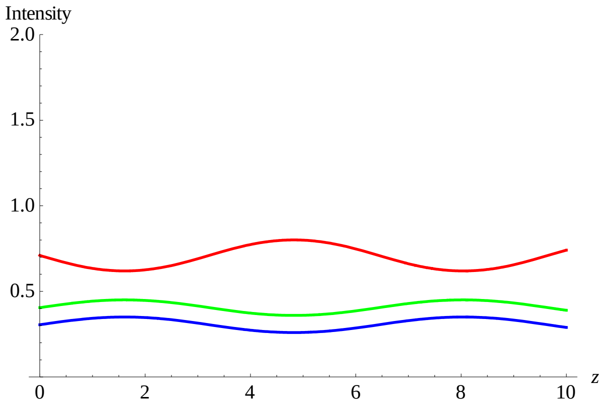

Fig 2 Periodic exchange of energy in the case of: $\Delta k = 0$; $c_1 = 0{,}1$; $c = 1{,}42$; $k = 0{,}1$.

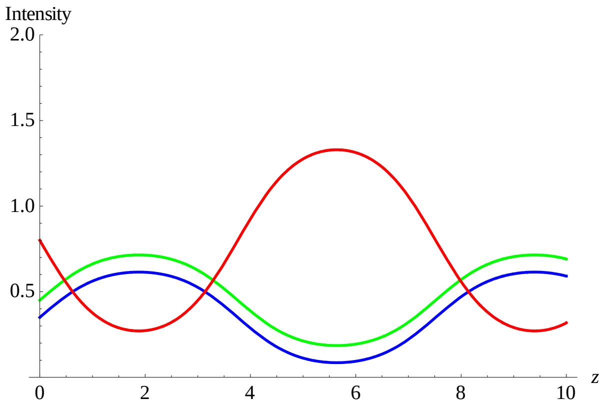

Fig 3 Periodic exchange of energy in the case of: $\Delta k = 0$; $c_1 = 0{,}1$; $c = 1{,}6$; $k = 0{,}53$.

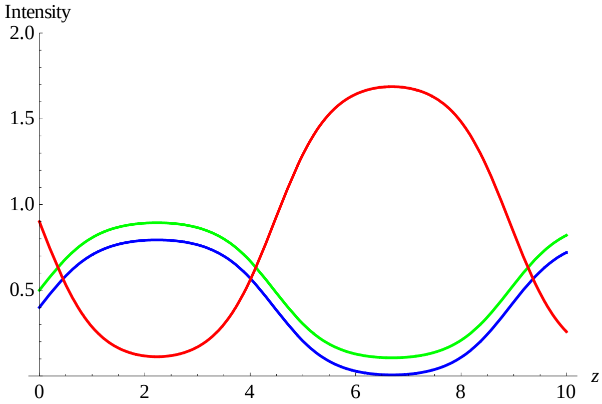

Fig 4 Periodic exchange of energy in the case of: $\Delta k = 0$; $c_1 = 0{,}1$; $c = 1{,}8$; $k = 0{,}79$.

In the case when $\Delta k \neq 0$, k^2 depends on the initial generalized phase $\psi(0)$. The periodic energy exchange between signal, idler and pump waves for the different values of the constants is presented on the figures below. For small values of $\Delta k \neq 0$ the energy exchange between the three waves is still well observed. If $\Delta k < 1$ the behavior of the waves is similar to the cases when $\Delta k = 0$.

On Fig. 5, Fig. 6 and Fig. 7 the constant Δk is equal to 1.

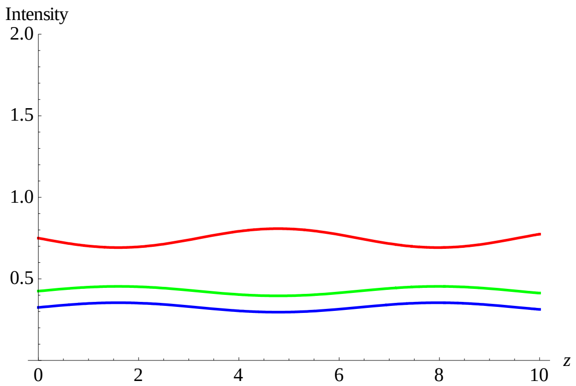

Fig 5 Periodic exchange of energy in the case of: $\Delta k = 1$; $c_1 = 0{,}1$; $c = 1{,}5$; $\psi(0) = 0$, $k = 0{,}057$.

As a result, we can assume that the energy exchange strongly depends on the value of the initial generalized phase $\psi(0)$. In the case of $\Delta k = 1$ the exchange of energy between the three waves is the most intensive when $\psi(0) = \pi / 2$.

On Fig. 8 and Fig. 9 are presented the graphics of energy exchange between pump, signal and idler waves in the case if $\Delta k = 1{,}5$.

On the Fig. 10 below it is presented the case when $\Delta k = 2$ and $\psi(0) = 0$.

When the values of $\Delta k > 2$ the condition for the parameter k $(1 > k^2 > 0)$ is not satisfied and the analytical solutions (17) don't describe the process of the FWM properly.

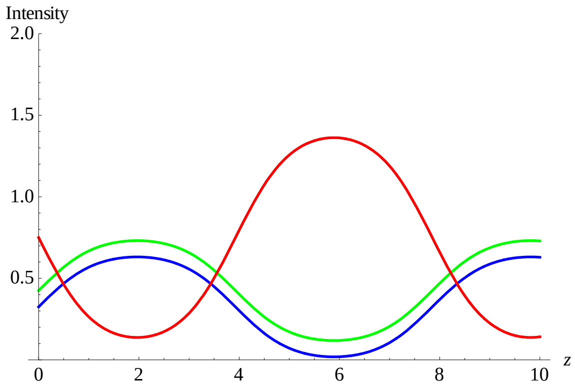

Fig 6 Periodic exchange of energy in the case of: $\Delta k = 1$; $c_1 = 0{,}1$; $c = 1{,}5$; $\psi(0) = \pi / 2$, $k = 0{,}61$.

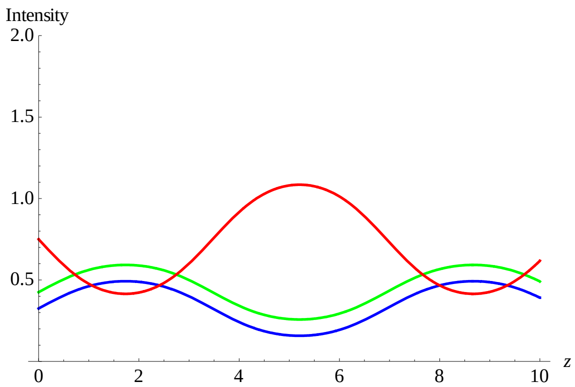

Fig 7 Periodic exchange of energy in the case of: $\Delta k = 1$; $c_1 = 0{,}1$; $c = 1{,}5$; $\psi(0) = \pi / 2$, $k = 0{,}61$.

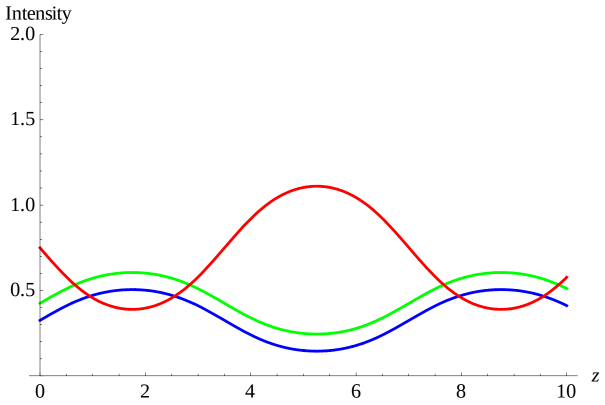

Fig 8 Periodic exchange of energy in the case of: $\Delta k = 1{,}5$; $c_1 = 0{,}1$; $c = 1{,}5$; $\psi(0) = 0$, $k = 0{,}36$.

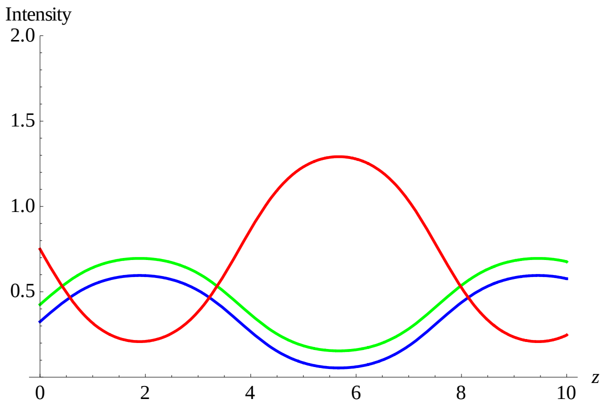

Fig 9 Periodic exchange of energy in the case of: $\Delta k = 1{,}5$; $c_1 = 0{,}1$; $c = 1{,}5$; $\psi(0) = 0$, $k = 0{,}36$.

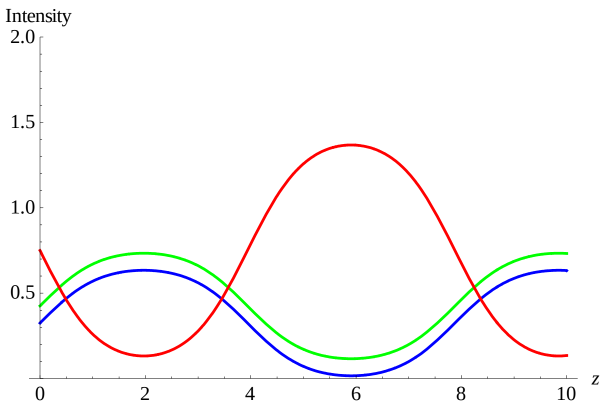

Fig 10 Periodic exchange of energy in the case of: $\Delta k = 2$; $c_1 = 0{,}1$; $c = 1{,}5$; $\psi(0) = 0$, $k = 0{,}62$.

4. CONCLUSIONS

We have shown a number of numerical calculations, based on the exact analytical solutions, found in (Andreeva et al), which describe the energy exchange between three optical waves through the process of FWM. They are presented in the form of Jacobi's elliptic function sine.

A connection between Jacobi's parameter k, the initial intensities of the waves and Δk was presented. This process is defined by the initial values of the constants Δk, c, c_1 and $\psi(0)$ which define the parameter k $(1 > k^2 > 0)$.

The obtained result and the observed effects in the present work are directly related to modern communication systems and optical sensors. That's way it is important to be studied in details.

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