Some Classical Solutions of the Pulsating String in Schrödinger Spacetime

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Abstract. In this paper we study a class of classical relativistic pulsating string solutions on five dimensional Schrödinger space times five dimensional sphere. This background is obtained by applying TsT transformations to the maximally symmetric $AdS_5 \times S^5$ space-time. After the transformations the new geometry acquires non-trivial antisymmetric B-field, which is the string analogy of a magnetic field. We also consider the standard expression for the Polyakov string action and derive the corresponding string equations of motion. Finally, we find some explicit solutions of the equations and analyze their properties.

Keywords: string theory, holography, Schrödinger spacetime, pulsating strings.

1. INTRODUCTION

In 1998 Juan Maldacena conjectured duality between string theory living in the bulk spacetime, and a lower-dimensional gauge theory living on the boundary of that region (Maldacena, 1998; Gubser, 1998; Witten, 1998). More precisely, the correspondence relates the quantum physics of strongly correlated systems to the classical dynamics of gravity in higher dimensions.

The correspondence is realized by matching between the local fields in the gravity theory and the field operators in the quantum theory. In other words, the sources ϕ_i in the dual gravity theory should match the tensor structure of the corresponding dual operator $\hat{O}_i$ in the field theory. Therefore, a scalar field can be dual only to a scalar operator, a vector field A_μ is dual to a current $\hat{J}_\mu$, while a spin-two field $g_{\mu\nu}$ is dual to a symmetric second-order tensor $\hat{T}_{\mu\nu}$. The latter can be naturally identified with the energy-momentum tensor of the fields in the quantum theory (Ramallo, 2015).

Although its impressive achievements the AdS/CFT correspondence on $AdS_5 \times S^5$ can naturally be generalized to include less supersymmetric backgrounds. The latter are more interesting from physical point of view, since they allow us to construct more realistic gauge theories, such as quantum chromodynamics. There are several ways to break the amount of supersymmetry in a given theory. One of them is to affect a change in the original AdS space with a suitable deformation, which will also violate the symmetries in the dual CFT. This way, one can produce new highly non-trivial gravitational backgrounds, which do not retain the symmetries of the original theory. One fine example is the five dimensional Schrödinger space, which is obtained by applying T-duality–shift–T-duality (TsT) type of transformations on the coordinates of the AdS_5 space. In this case, the dual CFT is invariant under a nonrealtivistic conformal group, known as the Schrödinger group. Moreover, the new Schr/CFT duality can be used to describe strongly correlated non-relativistic systems.

This paper is organized as follows. In Section 2 we obtain the five dimensional Schrödinger space by applying TsT transformations on the coordinates of the maximally symmetric AdS_5 space. In Section 3 we consider the standard Polyakov string action and derive the corresponding string equations of motions. Subsequently we solve the equations by imposing the Virasoro constraints and the ansatz for the relativistic pulsating strings. Finally, in Section 4, we briefly comment on our results.

2. SCHRODINGER GEOMETRY

The $Sch_5 \times S^5$ spacetime is a solution of type IIB string theory, obtained by applying TsT transformations on $AdS_5 \times S^5$ (Georgiou, 2017; Ouyang, 2017; Ahn, 2018).

The metric of the 5-dimensional AdS in global coordinates is given by

$$\frac{ds^2_{AdS}}{l^2} = -\left(1 + \frac{\vec{X}^2}{Z^2}\right) dT^2 + \frac{2dTdV + d\vec{X}^2 + dZ^2}{Z^2} \tag{1}$$

The TsT transformation is defined as follows. First we pick a $U(1)$ isometry direction on the S^5, corresponding to rotations along an angular direction χ, and one isometry direction on the AdS_5, which in this case is T. Next, we perform the following set of transformations (Guica, 2017):

- a T-duality along χ,
- a shift $T \to T + \hat{\mu}\tilde{\chi}$, where $\tilde{\chi}$ is the T-dual coordinate to χ,
- a T-duality back along χ.

Let us apply this set of transformations to the $AdS_5 \times S^5$ metric $ds^2 = ds^2_{AdS_5} + ds^2_{S^5}$, where the Kalb-Ramond 2-form field $B_{(2)} = 0$. We write the metric on S^5 in the form of $U(1)$ Hopf fiber over $\mathbb{CP}^2$,

$$ds^2_{S^5} = (d\chi^2 + P)^2 + ds^2_{\mathbb{CP}^2}, \tag{2}$$

$$ds^2_{\mathbb{CP}^2} = d\mu^2 + \sin^2\mu(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\sigma_3^2), \tag{3}$$

where $P = \sin^2\mu\sigma_3$ and σ_i are given by

$$\sigma_1 = \frac{1}{2}(\cos\alpha d\theta + \sin\theta\sin\alpha d\phi), \tag{4}$$

$$\sigma_2 = \frac{1}{2}(\sin\alpha d\theta - \sin\theta\cos\alpha d\phi), \tag{5}$$

$$\sigma_3 = \frac{1}{2}(d\alpha + \cos\theta d\phi), \tag{6}$$

After the TsT transformations we derive the Sch_5 part of the new metric:

$$\frac{dS^2_{Sch5}}{l^2} = -\left(1 + \frac{\hat{\mu}^2}{Z^4} + \frac{\vec{X}^2}{Z^2}\right) dT^2 + \frac{2dTdV + d\vec{X}^2 + dZ^2}{Z^2}, \tag{7}$$

while the S^5 part of the metric

$$\frac{dS^2_{S^5}}{l^2} = d\chi^2 + d\mu^2 + \frac{1}{4}\sin^2\mu(d\alpha^2 + d\theta^2 + d\phi^2) + 2\frac{\sin^2\mu}{2}d\chi d\alpha + 2\frac{\sin^2\mu\cos\theta}{2}d\chi d\phi + 2\frac{\sin^2\mu\cos\theta}{4}d\alpha d\phi, \tag{8}$$

is left unchanged. Moreover, after the transformation the new background picks up a non-zero anti-symmetric B-field:

$$B_{(2)} = l^2\frac{\hat{\mu}}{\alpha' Z^2}dT \wedge \left[d\chi + \frac{\sin^2\mu}{2}d\alpha + \frac{\sin^2\mu\cos\theta}{2}d\phi\right]. \tag{9}$$

3. CLASICAL SOLUTIONS

The Polyakov string action on curved background with metric G_{MN} is given by

$$S_P = -\frac{T}{2}\int d\tau d\sigma\left(\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN} - \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N B_{MN}\right), \tag{10}$$

where T is the string tension, $X^M = X^M(\tau,\sigma)$ are the string embedding functions, (τ,σ) are the coordinates on the string worldsheet, $h^{\alpha\beta} = \mathrm{diag}(-1,1)$ is the worldsheet metric, and $\epsilon^{\alpha\beta}$ is totally antisymmetric $\epsilon^{01} = -\epsilon^{10} = 1$. After varying the action, we end up with ten Lagrange-Euler equations, one for each global embedding coordinate X^M:

$$\frac{d}{d\tau}\left[\frac{\partial\mathcal{L}}{\partial\dot{X}^M}\right]+\frac{d}{d\sigma}\left[\frac{\partial\mathcal{L}}{\partial X'^M}\right]-\frac{\partial\mathcal{L}}{\partial X^M}=0, \quad (11)$$

and two Virasoro constraints

$$G_{MN}\left(\dot{X}^M\dot{X}^N+X'^M X'^N\right)=0, \qquad G_{MN}\dot{X}^M X'^N=0. \quad (12)$$

In order to solve the equations of motion and the Virasoro constraints, one has to impose a certain ansatz. Here, we are going to consider the following pulsing string ansatz:

$$T=\kappa\tau,\quad V=0,\quad \vec{X}=\vec{0},\quad Z=const\neq 0,$$
$$\chi=n_\chi\sigma,\quad \mu=\mu(\tau),\quad \theta=\theta(\tau),$$
$$\alpha=n_\alpha\sigma,\quad \phi=n_\phi\sigma,\quad n_i\in\mathbb{Z}. \quad (13)$$

Now, we find that the second Virasoro equation is satisfied, whereas the first one leads to

$$\dot{\mu}^2+\frac{1}{4}\sin^2\mu\left(\dot{\theta}^2+n_\alpha^2+n_\phi^2+4n_\chi[n_\alpha+n_\phi\cos\theta]+2n_\alpha n_\phi\cos\theta\right)-\left(1+\frac{\hat{\mu}^2}{Z^4}\right)\kappa^2+n_\chi^2=0. \quad (14)$$

The equations for $V,\vec{X},\chi,\alpha$ and ϕ are identically satisfied $(0=0)$, whereas the equations for T, Z, θ and μ yield

Along T:

$$\frac{d}{d\tau}\left[\sin^2\mu\left(n_\alpha+n_\phi\cos\theta\right)\right]=0 \quad\Rightarrow\quad \sin^2\mu\left(n_\alpha+n_\phi\cos\theta\right)=A=const. \quad (15)$$

Along Z:

$$Z^2=\frac{2\hat{\mu}^2\kappa}{b\left(2n_\chi+A\right)}. \quad (16)$$

Along θ:

$$\frac{d}{d\tau}\left[\sin^2\mu\dot{\theta}\right]-n_\phi\sin^2\mu\sin\theta\times\left[2n_\chi+n_\alpha-\frac{2b\kappa}{Z^2}\right]=0. \quad (17)$$

Along μ:

$$\frac{d}{d\tau}\left[2\dot{\mu}\right]+\frac{1}{2}\sin\mu\cos\mu\times[-\dot{\theta}^2+n_\alpha^2+n_\phi^2+4n_\chi\left(n_\alpha+n_\phi\cos\theta\right)+2n_\alpha n_\phi\cos\theta-\frac{4b\kappa}{Z^2}(n_\alpha+n_\phi\cos\theta)]=0. \quad (18)$$

Here $b=\hat{\mu}/\alpha'$ is a parameter. Obviously n_χ and A can not be zero at the same time, because $2n_\chi+A\neq 0$ must be fulfilled. This suggests the following cases

$$(i)\quad n_\chi\neq 0,\; A\neq 0,$$
$$(ii)\quad n_\chi=0,\; A\neq 0,$$
$$(iii)\quad n_\chi\neq 0,\; A=0,\ \text{such as or}\ n_\alpha+n_\phi\cos\theta=0,\ \text{or}\ n_\alpha=n_\phi=0. \quad (19)$$

Let's focus on the case (i). Replacing $-1\leq u=\cos\theta\leq 1$, one arrives at the equation

$$\left(\frac{d}{d\tau}u\right)^2=2(1-u^2)P_2(u), \quad (20)$$

where

$$P_2(u)=Cn_\phi^2u^2+n_\phi(2Cn_\alpha+K)u+n_\alpha(Cn_\alpha+K). \quad (21)$$

Here $K=2n_\chi+n_\alpha-\frac{2b\kappa}{Z^2}$, while $P_2(u)\geq 0$, $\forall\, u\in[-1,1]$. The case (ii) is similar to (i), but with different constant $K'=n_\alpha-\frac{2b\kappa}{Z^2}$. The discriminant is non-negative $D=K^2n_\phi^2\geq 0$ and consequently the equation for u takes the form

$$\left(\frac{d}{d\tau}u\right)^2=2Cn_\phi^2(1-u^2)(u-u_1)(u-u_2)\geq 0. \quad (22)$$

Depending on the signs of C and K and also on the order of the roots $-1,1,u_1,u_2$ of the right side of the equation, we have several cases of periodic solutions.

For example, at $C>0$ and $K>0$, the following cases are possible:

$$u_1<u_2<-1\leq u(\tau)<1, \quad (23)$$

$$u_1 < -1 < u_2 \le u(\tau) < 1, \qquad (24)$$

$$-1 \le u(\tau) < u_1 < u_2 < 1, \qquad (25)$$

$$-1 \le u(\tau) < 1 < u_1 < u_2. \qquad (26)$$

Let us consider the case $C > 0, K > 0$ and $u_1 < u_2 < -1 \le u(\tau) < 1$. Then $C = |C|, |u_{1,2}| > 1$ and we can integrate the equation (23). According to (Gradshteyn and Ryzhik, 2007), the solu-tion can be expressed as a first kind elliptic integral

$$\int_b^{u(x)} \frac{dz}{\sqrt{(a-z)(z-b)(z-c)(z-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\xi, r), \qquad (27)$$

where

$$\xi = \arcsin\sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}}, \quad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}, \qquad (28)$$

and

$$F(\xi, \mathrm{r}) = \frac{\sqrt{2|\mathrm{C}|\, \mathrm{n}_\phi^2 (\mathrm{a}-\mathrm{c})(\mathrm{b}-\mathrm{d})}}{2}\tau,$$
$$a = 1,\ b = -1,\ c = u_2 = -|u_2|,\ d = u_1 = -|u_1|. \qquad (29)$$

The solution of this equation is given by

$$\theta(\tau) = \arccos\frac{J}{H}, \qquad (30)$$

where

$$J = 1 - \frac{2|u_1|}{|u_1|-1} sn^2\left[\frac{\sqrt{2|C|n_\phi^2(|u_1|-1)(1+|u_2|)}}{2}\tau; r\right],$$
$$H = 1 + \frac{2}{|u_1|-1} sn^2\left[\frac{\sqrt{2|C|n_\phi^2(|u_1|-1)(1+|u_2|)}}{2}\tau; r\right],$$

Now we consider $(iii)\, n_\chi \ne 0, A = 0$. The first solution of $A = \sin^2\mu\left(n_\alpha + n_\phi\cos\theta\right) = 0$ is

$$n_\alpha + n_\phi \cos\theta = 0 \ \Rightarrow$$
$$\cos\theta = -\frac{n_\alpha}{n_\phi}, \quad \theta = const \qquad (31)$$

The equation of motion for μ takes the form

$$\dot{\mu}^2 = B^2 - \frac{1}{4}\sin^2\mu\left[n_\phi^2 - n_\alpha^2\right]. \qquad (32)$$

Comparing it with the non-zero Virasoro con-straint we find the constant $B^2 = \left(1 + \frac{\hat{\mu}^2}{Z^4}\right)\kappa^2 - n_\chi^2$. Now we can integrate

$$\int_0^\mu \frac{d\mu}{\sqrt{1-k^2\sin^2\mu}} = F(\mu, k) = \pm|B|\int_0^\tau d\tau, \qquad (33)$$

where $k^2 = \frac{n_\phi^2 - n_\alpha^2}{4B^2}$. Finally, one has $\sin\mu = sn(\pm|B|\tau, k)$, or

$$\mu(\tau) = \arcsin\left[sn(\pm|B|\tau, k)\right]. \qquad (34)$$

The second solution for $A = \sin^2\mu \times\left(n_\alpha + n_\phi\cos\theta\right) = 0$ is $n_\alpha = n_\phi = 0$. From the equations of motion and the Virasoro con-straint we find the following equation for μ:

$$\frac{d^2}{d\tau^2}[\cos\mu] + B^2\cos\mu = 0. \qquad (35)$$

The solution to this equation is

$$\mu(\tau) = \arccos\left[C_1\cos(B\tau) + C_2\sin(B\tau)\right]. \qquad (36)$$

4. CONCLUSION

Our study is instigated by the remarkable success of string theory in the context of the celebrated AdS/CFT correspondence. This mo-tivates us to consider various string confi-gurations on curved spacetime backgrounds, in hope to reveal new and important aspects of the theory.

In recent years great interest attract spacetime geometries with reduced or absent superconformal symmetry, since they allow us to construct more realistic gauge theories, such as quantum chromodynamics. One fine exam-ple is the five dimensional Schrödinger space, which is obtained by applying T-duality-shift-T-duality (TsT) type of transformations on the coordinates of the AdS_5 space. In this case the dual CFT is invariant under a non-relativistic conformal group, known as the Schrödinger group.

In this report we have consider a particular string configuration, namely the pulsating string on the Schrödinger space. We have derived the corresponding equations of motion by varying the Polyakov string action. Consequently we solved them in terms of elliptic functions.

Our future investigations will be focused on the quantum aspects of the theory and uncovering interesting features about the dual non-relativistic gauge theory.

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