Heuristic Physics Problems in the Works of George Pólya

Kalin Angelov[1] and Fisnik Hoja[2]

Sofia University St. Kliment Ohridski - Faculty of Physics, 5 J. Baucher blvd., Sofia, Bulgaria

Abstract. George Pólya (1887-1985) is considered the founder of modern heuristics research in natural-mathematical sciences. By developing heuristic methods for solving problems, mainly in the field of mathematics, he also provides a number of physical interpretations as well as a number of interesting physics tasks from the mechanics, optics, etc. sections. In this article, we examine a corpus of these problems, included in his works: "Mathematical Discovery", "How to Solve It" and "Mathematics and Plausible Reasoning".

Keywords: Heuristics, heuristic problems, physical problems, physical mathematics, methodology of physics education, pedagogy, psychology.

1. INTRODUCTION

George Pólya (1887-1985 is a world-renowned Hungarian mathematician who has left lasting traces in various courses of mathematics and the teaching of natural-mathematical sciences. His significant contributions are in the fields of mathematical analysis, algebra, probability theory and combinatorics, numerical theory, potential theory, mathematical physics, and the relationship between physics and geometry. In the words of Jean-Pierre Caen[i], the glossary of the mathematical terms associated with his name is impressive: the poi's curve, the Pólya's functions, Pólya's indica, Pólya's density, the Pólya-Shur functions, Pólya-Segyo's dimension, random wandering of Pólya, Pólya's theory.

Of particular interest and world-wide popularrity are his pedagogical developments and related heuristic methods and plausible reasoning on problem solving and the role of educational problems.

It is believed that Pólya is the founder of the modern definition of the term **heuristics**[ii]. The ideas put forward in his work in this field give new life and inspiration to the research related to the problems of the creative process and the methods for increasing its efficiency. Pólya's heuristics do not just influence but probably play a significant role in the development of expert systems in the field of artificial intelligence.

Once in Mathematical Society, were particularly popular, his books "How to Solve a Problem", "Mathematical Discovery", "Mathematics and Reasonable Thoughts" - p. 1 "Induction and analogy in mathematics" and p. 2 "Schemes of plausible conclusions". He has written seven books and over 250 articles. He teaches combinatorics at the age of 90.

Like many mathematicians from the mid-20th century, Pólya is also tempted by physics. Not by accident one of his significant works (with G. Szegő) is Isoperimetric Inequalities in Mathematical Physics, George Pólya, Gábor Szegő - Princeton University Press, 1951. However, the main corpus of problems in Polya's pedagogical development are mathematical, but there are also many interesting physical problems. In this publication we will show some particularly interesting examples of physical problems with a strong heuristic element in their decisions. Three of them will look in detail, and another five briefly will be described.

[1]Kalin Angelov - kalin_angelov_mat@abv.bg
[2]Fisnik Hoja - fisnikahoxha@yahoo.com

The examples are taken from the books by George Pólya

1. "How to Solve It"[iii].
2. "Mathematical Discovery".
3. Mathematics and Plausible Reasoning - Vol. 1: Induction and Analogy in Mathematics

2. THREE HEUROISTIC PROBLEMS WITH DETAILED ANALYSIS.

Problem 2. 1. "An example of physics [iv]"

A iron sphere floats in mercury. Above mercury, pour water until it covers the sphere. Will the sphere sink, rise or remain on the same level?

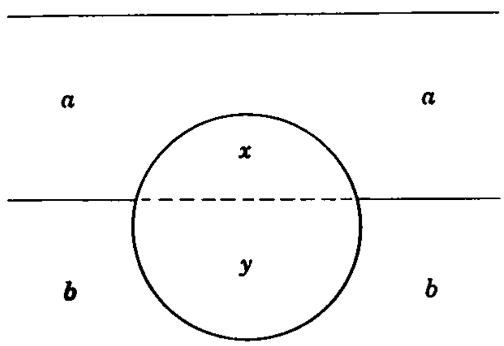

Fig. 1

Let's compare two situations. At both the lower part of the sphere it is submerged in mercury. In the first case, the top of the sphere is in the air, and in the second - in the water.

In what situation is the upper part (above the mercury level) greater?

According to Pólya, this is a purely qualitative question, but we can also give him a quantitative statement:

Calculate the portion of the sphere volume that is above the mercury level in both positions.

By itself, the transition from a qualitative to a quantitative problem is a heuristic skill that, if done quickly and logically correct, leads to faithful, scientifically grounded conclusions, shortens time, and protects against erroneous conclusions. In this case, we have a good example of such a transition.

Pólya's heuristic (plausible reasoning) is as follows:

The qualitative question gives a plausible response with purely intuitive considerations, imagining a ***continuous transition*** from one situation to another.

Imagine that over the mercury is a fluid that covers the top of the sphere *and changes its density continuously*. Let this imaginary fluid have at first a density of zero (vacuum). Then its density increases, soon it equals the density of the air and then the density of the water. "If you still do not see how this change works on the floating ball, let the *density grow even more*". When the density of the imaginary fluid equals that of the iron, the sphere will rise above the mercury. Indeed, if the density increases even further, the sphere will float into the imaginary fluid and rise just above it.

(2) Quantitative decision:

It requires the values of the three specific weights and the knowledge of Archimedes' law.

Introduce the following weights for specific weights and volume proportions[v]:

$a = 1,0$ - the specific weight of the water (or the fluid above the mercury and the iron sphere);

$b = 13,6$ - the specific weight of mercury;

$c = 7,84$ - the specific weight of the iron;

v - the volume of the iron sphere;

x - the part of the sphere over the mercury;

y - the part of the sphere under the mercury;

We obtain a system of two equations with two unknowns:

$$\begin{vmatrix} x + y = v \\ ax + by = cv \end{vmatrix}$$ - the Archimedes' law;

The solutions of the system are:

$$x = \frac{b-c}{b-a} v$$

and

$$y = \frac{c-a}{b-a} v$$

And so:

- if there is a vacuum above the mercury, then $a = 0$ and $x_{\text{vacuum}} = 0{,}423v$;
- if there is water above the mercury, then $x_{\text{water}} = 0{,}457v$;

$$x_{\text{water}} > x_{\text{vacuum}}$$

which is consistent with the conclusion of intuittive considerations.

The solutions of the system “materialize” the intuitive reasoning.

Problem 2. 2 “Physical Mathematics”[vi].

We will look at a task on “Optical Interpretation” in geometry, which will illustrate Pólya’s assertion that often *Mathematical problems are often inspired by nature, or rather by our interpretation of nature, the physical sciences. Also, the solution of a mathematical problem may be inspired by nature physics provides us with clues with which, left alone, we had very little chance to provide ourselves. Our outlook would be too narrow without discussing mathematical problems suggested by physical investigation and solved with the help of physical interpretation.*[vii]

From examining the distribution of light we come to the following purely geometric task:

Given two points and a straight line) all in the same plane, both points on the same side of the tine. On a given straight line, find a point such that the sum ofits distances from the two given points be a minimum.

l, A and B denote the two given points, the given straight line l, X a variable point of the line.

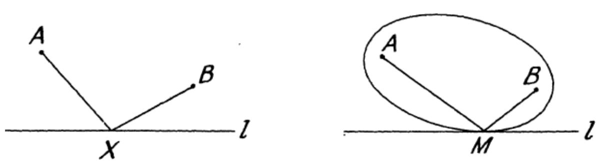

Fig. 2

The solution achieved by minimizing the sum of segments using the knowledge of the properties of the ellipse is:

The desired minimum is attained at the point of contact of the prescribed path l with an ellipse the foci of which are the given points A and B - Fig. 2.

Nature suggests a solution:

The physical treatment of the task could be:

Point A is the light source, the eye of the observer, l - the reflecting plane (water surface, mirror) - Fig. 3.

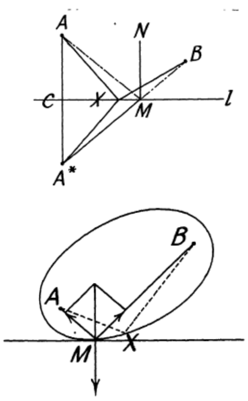

Fig. 3

Our physical experience suggests that we introduce the point $A*$, which is a mirror image of A.

The heuristic moment, according to George Pólya, is the “introduction” of $A*$, which he expresses with the words: *“his point $A*$ changes the face of the problem. We see a host of new relations”* (Fig. 3).

The solution is obvious:

$AX = A*X$, we examine the equivalence of the triangles ΔACX and $\Delta A*CX$, l is symmetral, $\Rightarrow$ $AX + XB = A*X + XB$ - *Yet the right-hand side is obviously a minimum when $A*$, X and B are on the same straight line. The straight line is the shortest. This is the solution. The point M, the minimum position of X, is obtained as intersection of the line / and of the line joining $A*$ and B* - Fig. 3.

Introducing $MN \perp l$, equality $\sphericalangle AMN = \sphericalangle BMN$ characterizes both the shortest path and the equilibrium between the angle of fall and the angle of reflection (in today's traditional denominations $\alpha_1 = \alpha_2$). The reflected light beam selects the shortest possible route between the object and the eye, a fact established by Heron of Alexandria (about 10 - 75 y.) and later compiled by Pierre de Fermat (1601-1665) as the principle of geometric optics.

Problem 2. 3. *„Reinterpretation”*[viii] (logical continuation of the Problem 2. 1).

In this context, George Pólya takes a close look at an interesting proof of Snellius' law, a heuristic analogy of mechanical "reinterpretation" to an optical problem.

When switching from water to air, the light does not follow the "shortest possible route" principle but describes a broken line AXB - Fig. 4.

The first basic heuristic reasoning for light diffusion is: "Light has different speeds in water and air and follows the principle of minimality in any different environment."

Second heuristic reasoning:

"As the light moves at steady speed in choosing the shortest course, she also chooses *the fastest rate*."

Third heuristic reasoning:

"If the velocity depends on the medium traversed, the shortest course is no more necessarily the fastest. Perhaps the light chooses always the fastest course, also in proceeding from the water into the air."

This train of ideas leads to a clear problem of minimum (Fig. 4):

Given two points A and B, a straight line l separating A from B, and two velocities

u and v, find the minimum time needed in travelling from A to B; you are supposed to travel from A to l with the velocity u and from l I to B with the velocity v.

The essence of the problem is to find the point X in the above-mentioned problem, given u, v and l.

Because in each of the environments the motion of the light is even, the solution is to minimize the sum of the times of movement in each of them:

$$\frac{AX}{u} + \frac{XB}{v} \to \min$$

According to Pólya, the problem is not easy to solve without differential calculus, and *Fermat solved it by inventing a method that eventually led to the differential calculus.*

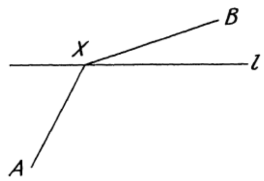

Fig. 4

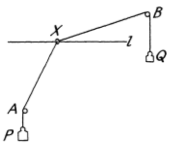

Fig. 5

The heuristic mechanism offered by George Pólyais through the following mechanical interpretation (Fig. 5):

Ring X can slide along a fixed horizontal rod l and through it there are threads with fixed lengths AX and XB (with reels in A and B), on which are mounted dumbbells p - in point P and q – in point Q, with the usual idealizations of rigidity, indolence, ignorance of resistances, and so on. The two dumbbells have different weights $(p \neq q)$ and strive to settle as low as possible, *"the potential energy of the system must be minimum"*, which, written in algebraic form, would look like this:

$$AP.p + BQ.q \to \min$$

$$AX.p + XB.q \to \min$$

bc. threads AX and XB are of fixed length

This mechanical task completely coincides with the optical one, if we lay down:

$$p=\frac{1}{u},\ q=\frac{1}{v}$$

The problem of balancing completely coincides with the problem of rapid deployment:

$$\frac{AX}{u}+\frac{XB}{v}\to \min$$

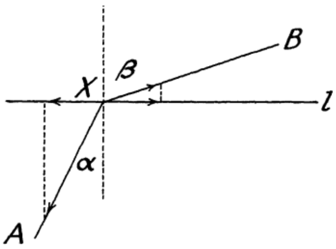

Fig. 6

Vertical reactions are not considered due to the rigidity of the rod on which the ring is located.

Then, in order to achieve the equilibrium of the system *with minimal potential energy*, the horizontal components of the response of the two stresses must be equal in size and opposite in the direction. This dependence is expressed by the introduction of angles α and β with respect to the perpendicular (in the optical context - the angles of fall and refraction) in point X (the horizontal vectors in red in Fig. 6) and we get:

$$\frac{1}{u}\sin\alpha=\frac{1}{v}\sin\beta \Rightarrow \frac{\sin\alpha}{\sin\beta}=\frac{u}{v}$$

This is the *condition of minimum*.

The essential for this type of heuristic solution is that the task at first had a physical (optical) interpretation, and to solve it we found another physical (mechanical) interpretation. Again, in Pólya's words: *Such solutions may reveal new analogies between different physical phenomena and have a peculiar artistic quality*[ix].

3. OTHER HEURISTIC PROBLEMS – DESCRIPTION

3. 1. Archimedes' discovery of the integral calculus[x].

In sphere volume studies, Archimedes reaches the integral calculus, guided by physical intuition, in his view "investigated some problems in mathematics by means of mechanics". The "mechanicity" of its method consists in the "suspension" in cylinder equilibrium and its sphere and cone.

3. 2. Graphic time table[xi].

In problems with several objects moving along the same trajectory, Pólya explains the methodical utility of introducing a rectangular coordinate system, constructing the motion charts of each body and their intersection points. This scheme is called Graphic time table.

3. 3. Newtonian problems[xii].

On the next few pages after the aforementioned problem, George Pólya offers us a few original problems from Isaac Newton – for the two postmen who are 59 miles away; to determine the depth of a well by the sound of falling stone; to determine the comet's course in three observed situations.

Problems primarily cause methodical, historical and aesthetic-artistic interest but also generate heuristic thought processes.

3. 4. A rate problem[xiii].

The condition of the problem is as follows:

Water is flowing into a conical vessel at the rate r. We are given the radius of the base and the height of the vessel. Find the rate of elevation at a given initial depth.

The decision was examined in detail in a commentary style similar to Socrates' dialogues. The benefits of credible plausible reasoning in the math and physics teaching process and the benefits of analytical experien-ce have been demonstrated.

3. 5. Test by dimension[xiv].

The problem of expressing the period of a simple (mathematical) pendulum is considered:

$$T = cl^m g^n$$

Reflecting on the dimensions of the multipliers, the formula is derived in the traditional form:

$$T = c\sqrt{\frac{l}{g}}$$

For the significance of “the dimensionality test”, we will literally quote Pólya: *The test by dimension is even more important in physics than in geometry. ... the consideration of the dimensions has allowed us to foresee quickly and with the most elementary means an essential part of a result whose exhaustive treatment demands much more advanced means. And this is so in many similar case.*

3. 6. Bright idea (or “good idea” or “seeing the light”)[xv]**.**

We will conclude with a historical-astronomical description of the *Bright idea* of the most popular work by George Pólya[xvi].

Pólya offers us an Aristotle definition of *“sagacity”*:

“Sagacity is a hitting by guess upon the essential connection in an inappreciable time. ... Or observing that the bright side of the Moon is always toward the sun, you may suddenly perceive why this is; namely, because the moon shines by the light of the sun.”.[xvii]

REFERENCES

Polya G., 1972. *How to Solve It*. Narodna prosveta, 146 pp. (full text in Bulgarian).

Polya G., 1968. *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, in 2 volumes, Narodna prosveta, 474 pp. (full text in Bulgarian).

Polya G., 1970. *Mathematics and Plausible Reasoning - Vol. 1: Induction and Analogy in Mathematics, Vol. 2: Patterns of Plausible Inference*, Narodna prosveta, 578 pp. (full text in Bulgarian).

Kahane J.-P., 1988. *La Grande Figure de George Polya*, Proceedings of the Sixth ICME, Budapest, pp. 79-100.

[i]Jean-Pierre Kahane (1926-2017) - Professor at the University of Paris and President of the International Commission on Mathematical Education (ICMI) at the International Mathematical Union (IMU).

[ii]**Heuristics** – “Heuristics or” ars inveniedi “was the name of a not very clearly defined field of study, belonging to logic, philosophy or psychology .... Modern heuristic endeavors to understand the process of solving problems, especially the mental operations typically useful in this process.”

[iii]The citations and pages mentioned are in Bulgarian translations, detailed in the cited literature, and the illustrations are copied from the original English-language editions with minimal additions on them. When the names of the tasks, their terms and their parts of the decisions are according to the original text, they are given in quotation marks or italics.

[iv]“Mathematical Discovery”.

[v]Markings are strictly in the original text.

[vi]From Mathematics and Plausible Reasoning - Vol. 1: Induction and Analogy in Mathematics, p.178, IX-Physical Mathematics.

[vii]Ibidem.

[viii]From Mathematics and Plausible Reasoning - Vol. 1: Induction and Analogy in Mathematics.

[ix]Ibidem.

[x]Ibidem – p. 191.

[xi]“Mathematical Discovery”.

[xii]Ibidem.

[xiii]„How to Solve It”.

[xiv]Ibidem.

[xv]Ibidem.

[xvi]By the end of the 1990s, „How to Solve” It was translated into seven languages with a circulation of more than one million.

[xvii]Poya notes that the text is slightly edited and offers us a more accurate translation reference with William Whewell, *The Philosophy of the Inductive Sciences* (1847), vol. II, p.131.